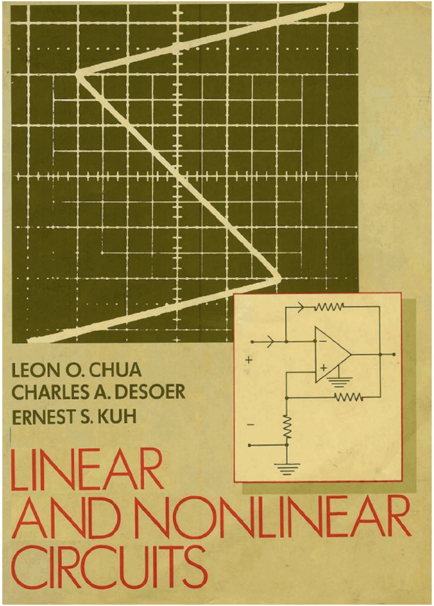
Splitting Algorithm for I/O Analysis of Negative Resistance Circuits

Amritam Das

Collaborators: Karl H. Johansson, Rodolphe Sepulchre, Thomas Chaffey

Negative Resistance Circuit: Inspiration



The fundamental device
for switches and oscillations
in the pre-digital age

Negative Resistance Circuit: Thematic Viewpoint

2.5 GAINS

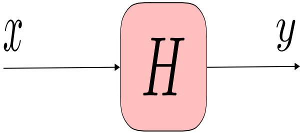
It is now desired to compute or bound the norm of the output of a system when that of the input is known. In order to do this, it is necessary to know the amplification of the system, which will differ, in general, for every input. We are especially interested in the largest incremental amplification that a system is capable of. The maximum incremental gain of an operator is therefore defined as the largest possible ratio of the norm of the difference between any two outputs to that between the inputs, that is

$$\text{incr}|||\underline{H}||| = \underset{x,y}{\text{l.u.b.}} \frac{\|\underline{H}(x)-\underline{H}(y)\|}{\|x-y\|} \tag{23}$$

*George Zames (1960): "Nonlinear Operations of System Analysis"

Thematic Viewpoint: Incremental I/O Analysis

- Let us look at one element:



- The supplied energy (in incremental sense) $\Delta V := \left\langle x_1 - x_2, y_1 - y_2 \right\rangle$
- H is **incrementally passive** if $\Delta V \geq 0$

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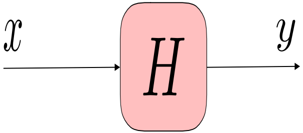

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In the language of operator theory

For causal operators, incremental passivity is synonymous to *Monotonicity* on signal space
(A positive change in the input should cause a positive change in the output)

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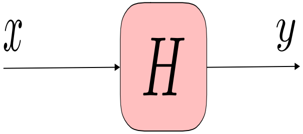

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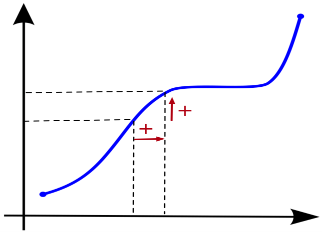

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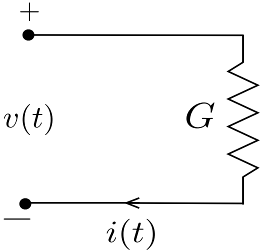
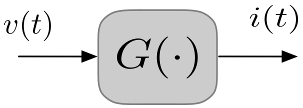
*Minty(1960), "Monotone networks"
*Camlibel and Schumacher (2016), "Linear passive systems and maximal monotone mappings"

Characterization of Negative Resistors Using Monotone Resistor





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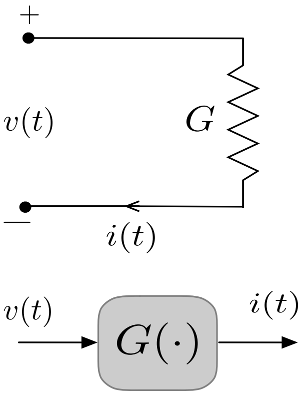

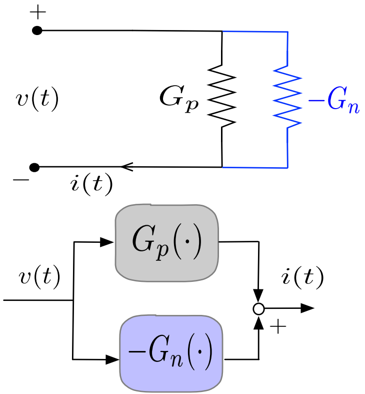

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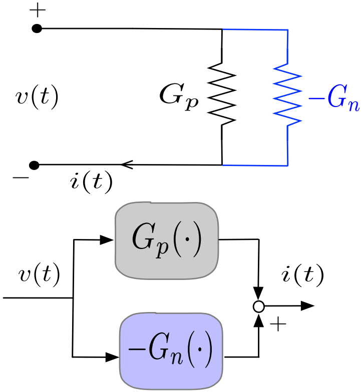
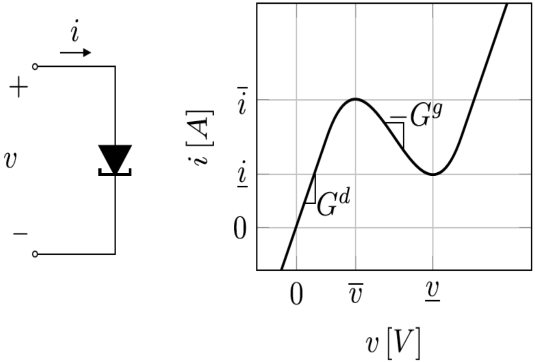
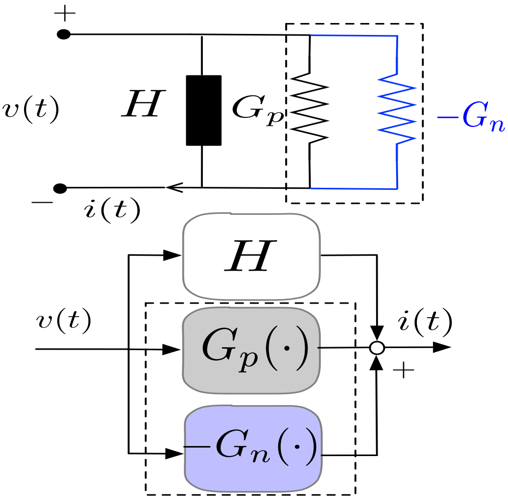




Tunnel Diode

$$i = \frac{v^3}{3} - v$$

Netgative Resistor to Negative Resistance Circuit



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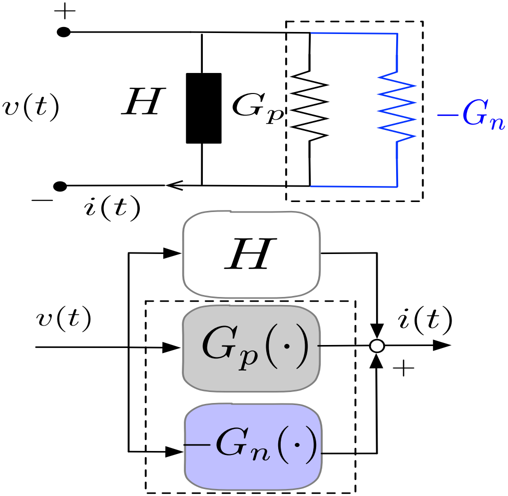

- H can be any linear passive transfer function
- Series, parallel and negative feedback interconnection preserves monotonicity

Note: For technical reason, H, G_p are assumed to be *maximal*

A Simple Illustration: Van der Pol Oscillator

Van der Pol Oscillator: $\ddot{v} + K(v^2 - 1)\dot{v} + v = 0$

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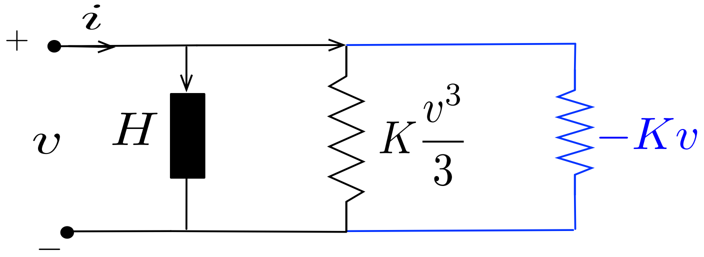
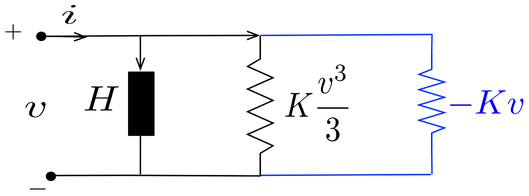


Mixed-Feedback Systems: A Simple Illustration



I/O relation of Van der Pol Oscillator

$$0 = \underbrace{Hv}_{\text{linear}} + \underbrace{K\frac{v^3}{3}}_{G_p\text{:static cubic}} - \underbrace{Kv}_{G_n\text{:static linear}} \quad , H(s) = \left(\frac{s^2+1}{s}\right)$$

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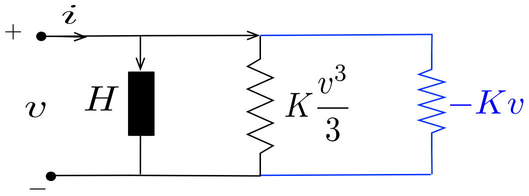

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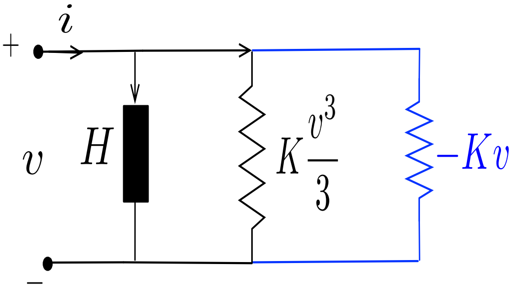
How do we characterize each constituent?

Circuit of Van der Pol: Passivity/Monotonicity View-Point



I/O relation of Van der Pol Oscillator

$$0 = \left(\frac{s^2+1}{s}\right)v + +K\frac{v^3}{3} - Kv$$

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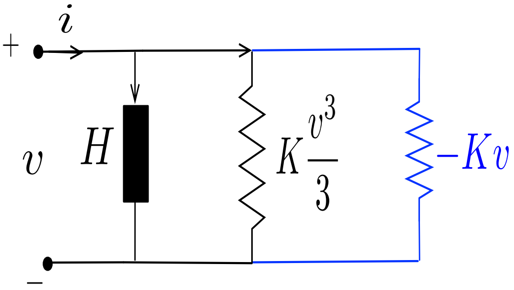

- *Characterize each operators based on monotonicity*

  Operator F is called monotone iff $\left\langle z - r, F(z) - F(r) \right\rangle \geq 0$ for any inputs $z, r \in \mathcal{X}$

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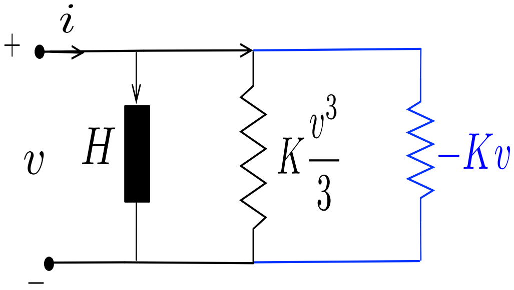

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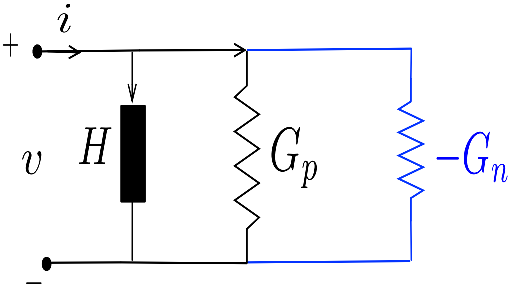
I/O relation is a difference of monotone relation

General Setting of Negative Resistance Circuit



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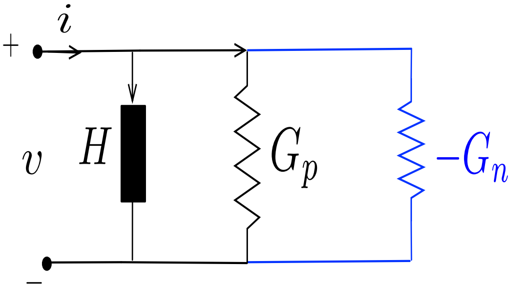


- H, G_p, G_n are monotone operators
- I/O relation: $0 = Hv + G_p(v) - G_n(v) - i$

Question

How to develop a computational (algorithmic) tool?

e.g. Given $i = i^*$, find $v \in \mathcal{V}$ such that $0 = Hv + G_p(v) - G_n(v) - i^\star$

Computational Frameowork: Finding Zero of Monotone Operators

Borrowed from optimization theory:

Given a montone operator R, a solution to the following problem

$$0 = R(x)$$

is a fixed point of the following relation

$$x_f = F(x_f), \quad F \text{ is derived from } R$$

You can find x_f by solving the fixed point iteration

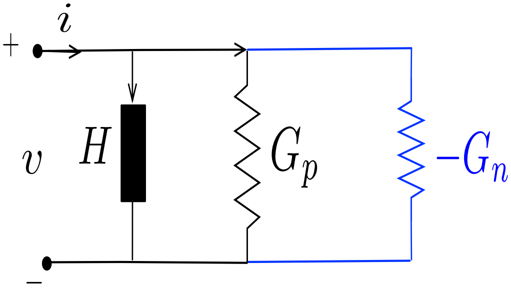
$$x_f^{k+1} = F(x_f^k)$$

If $F := (I + \alpha R)^{-1}$ (a.k.a resolvent), then the fixed point iteration converges. $\alpha > 0$ is the step-size

Advantages

- Cheap computation of resolvents for linear operator and some static NL functions
- Solving $0 = \sum_{i=1}^{n} R_i(x)$ can be split into the computation of each resolvent F_i

Computing Zero of Difference between Monotone Operators



$$0 = \underbrace{Hv + G_p(v) - i^\star}_{A(v)} \underbrace{- G_n(v)}_{B(v)}$$

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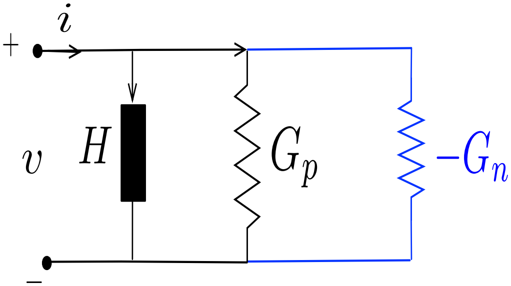
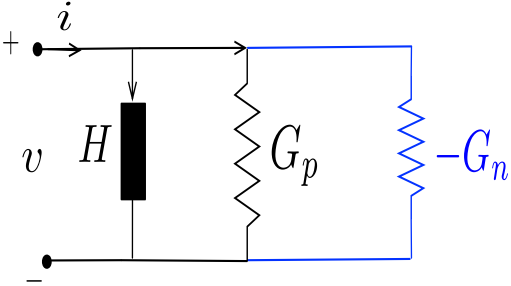


$$0 = \underbrace{Hv + G_p(v) - i^\star}_{A(v)} \underbrace{- G_n(v)}_{B(v)}$$

- **Algorithm:**
  1: Given: Initial guess v_0
  2: **Repeat for** i
  3:
  $$v_{i+1} = \operatorname*{find}_y 0 \in A(v) - B(v_i)$$
  $$i := i + 1$$
  4: **until** stopping criterion is satisfied

Operator Splitting

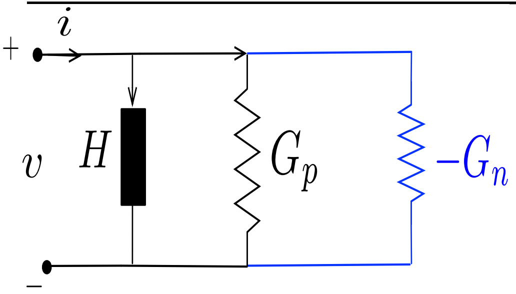


At i^{th} iteration, given v_i, solve for v_{i+1}

$$0 = \underbrace{Hv_{i+1}}_{F_1} + \underbrace{G_p(v_{i+1}) - i^\star - G_n(v_i)}_{F_2}$$

$$\implies 0 = F_1(v_{i+1}) + F_2(v_{i+1})$$

Operator Splitting



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- **Algorithm with Douglas Rachford Splitting:**

Given: Initial guess v_0

Repeat for i

  $y^0 = v_i$

  for $j = j + 1$ **do**

$$\begin{aligned} w^{j+1/2} &= \mathrm{res}_{F_1,\lambda}(y^j) \\ z^{j+1/2} &= 2w^{j+1/2} - y^j \\ w^{j+1} &= \mathrm{res}_{F_2,\lambda}(z^{j+1/2}) \\ y^{j+1} &= y^j + w^{j+1} - w^{j+1/2} \end{aligned}$$

  end for when stopping criterion met

$$v_{i+1} = y^{j+1}$$

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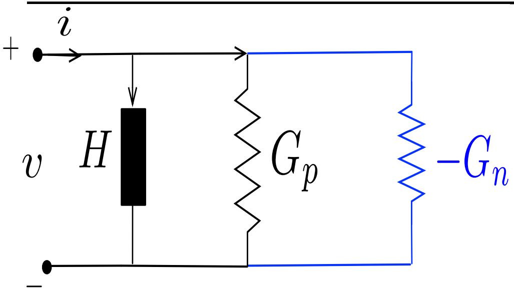
Operator Splitting



At i^{th} iteration, given v_i, solve for v_{i+1}

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$$\implies 0 = F_1(v_{i+1}) + F_2(v_{i+1})$$

Results in a locally convergent algorithm

- **Algorithm with Douglas Rachford Splitting:**

1: Given: Initial guess v_0
2: **Repeat for** i
3: $y^0 = v_i$
4: **for** $j = j + 1$ **do**

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5: **end for** when stopping criterion met

$$v_{i+1} = y^{j+1}$$

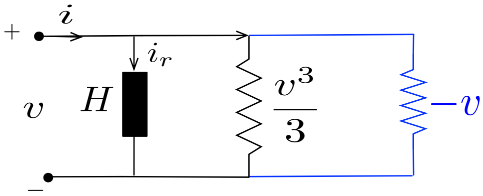
$$i := i + 1$$

6: **until** stopping criterion is satisfied

*Das, Chaffey, and Sepulchre (2022): Oscillations in Mixed-Feedback Systems, Automatica

Illustration: Fitzhug Nagumo Circuit

$$
\begin{aligned}
C\dot{v} &= -\frac{v^3}{3} + v - i_r + i, \\
L\dot{i}_r &= -Ri_r + v.
\end{aligned}
$$



$$
H(s) := \frac{LCs^2 + CRs + 1}{Ls + 1}
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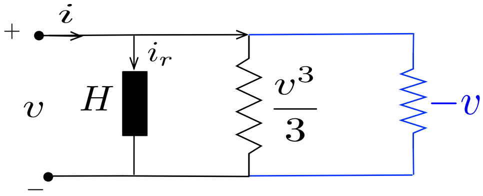

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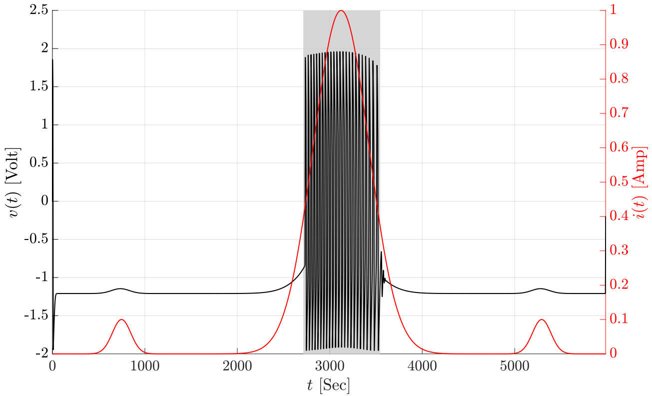

Wrap Up

Towards a computational framework of negative resistance circuits

What we observe

- Negative resistance circuit is a parallel interconnection of two montone one port element and one anti-monotone element
- I/O relation amounts to finding zeros of difference between two monotone operators
- Iterative splitting algorithm provides a comoputational framework for these circuits.

Open Problem: Can we exploit the connection of Monotone Operators and Convex Optimization?

- How do we infer stability of patterns?
- How to design robust oscillator (e.g. avoid hidden oscillation)?

Some relevant fields: Relay-feedback systems, Dominance theory, DC programming