Learning Flows of Control Systems

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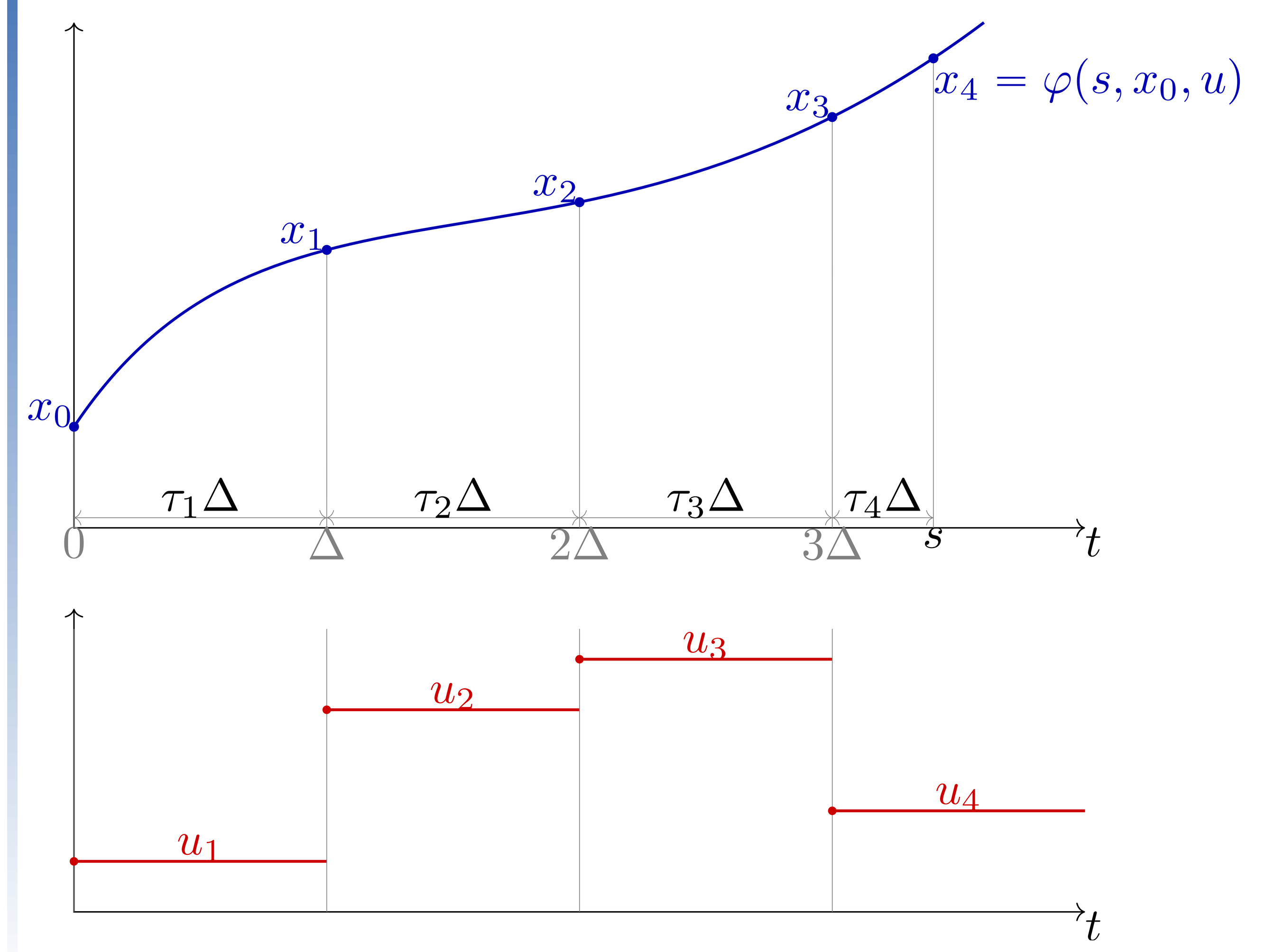
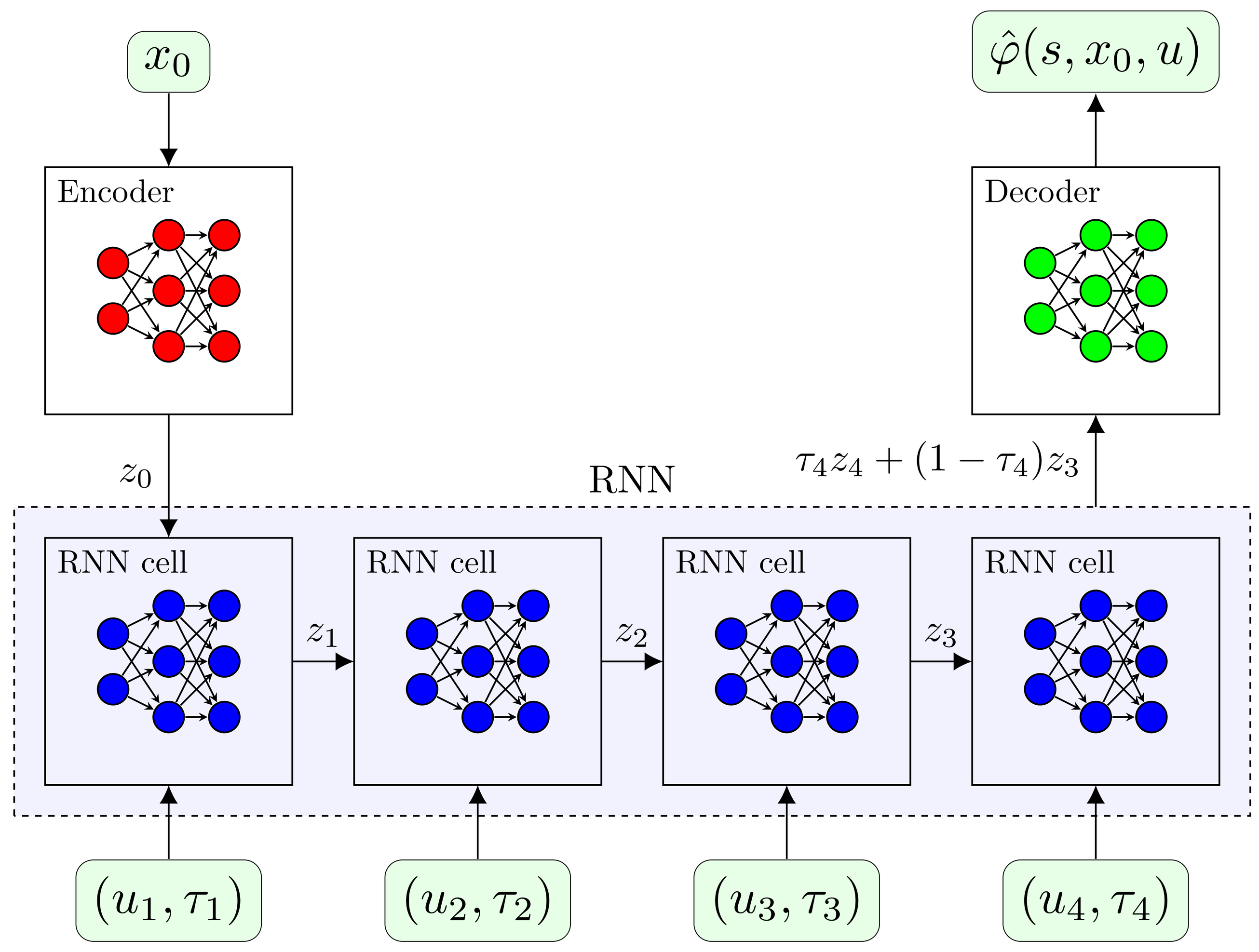
Main idea

- A control system Σ is defined as the quadruple: $\Sigma = (\mathcal{T}, \mathcal{X}, \mathbb{U}, \varphi)$.
- The flow dictates the evolution of states in Σ and is defined as a mapping $\varphi : \mathcal{T} \times \mathcal{X} \times \mathbb{U} \to \mathcal{X}$.
- We learn the flow φ by the following model architecture:





Class of systems

- Properties: They are considered to be *causal* and *time-invariance*.
- Type of inputs: For a given $\Delta > 0$, there exists a sequence $\{\theta_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^p$ for which the control input $u(t)$ is defined as

$$u(t) = \mathfrak{u}\big[\{\theta_k\}_{k\in\mathbb{N}}\big](t) := \sum_{k=1}^{\infty} \alpha\left(\theta_k, \frac{t}{\Delta}\right) \mathbf{1}_{[(k-1)\Delta, k\Delta)}(t),$$

and the set of controls $\mathbb{U}$ is defined as the image of $\mathfrak{u}$:

$$\mathbb{U} := \left\{\mathfrak{u}\big[\{\theta_k\}_{k\in\mathbb{N}}\big] : \theta_k \in \mathbb{R}^p\right\}.$$

Learning formulation

- Data from a control system is typically available in the following form

$$\left\{\left(\{t_k^i\}, x^i, u^i, \{\xi_k^i\}\right),\ k = 1, \ldots, K,\ i = 1, \ldots, N\right\},$$

where $\xi_k^i = \varphi(t_k^i, x^i, u^i)$, $t_k^i \in [0, T]$ is an increasing sequence.
- Learning problem amounts to searching for a minimiser $\hat{\varphi} \in \mathcal{H}$ of

$$\hat{\ell}_T(\hat{\varphi}) := \frac{1}{N}\sum_{i=1}^{N}\frac{1}{K}\sum_{k=1}^{K}\left\|\xi_k^i - \hat{\varphi}(t_k^i, x^i, u^i)\right\|^2.$$

Assumption: To capture the inter-sample behaviour, assume $\bigcup_i \{t_k^i\}_{k=1}^K \not\subset \{k\Delta\}_{k=0}^{\infty}$.

A key result due to causality

Observation

At any time during the first control period $[0, \Delta]$, we can define $\Phi : [0, 1] \times \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ such that

$$\Phi(\tau, x, \theta_1) := \varphi(\tau\Delta, x, u).$$

Through Φ, a finite-dimensional vector of parameters (as opposed to functions) directly maps to the flow.

Generalisation

For an arbitrary time instant $s > 0$, the flow $\varphi(s, x, u)$ can be computed as follows:

1. Construct a map $\mathrm{d}_\Delta : (s, u) \mapsto \{\tau_k, \theta_k\}_{k=1}^{k_s+1}$ such that

$$k_s := \lfloor s/\Delta \rfloor, \qquad \tau_k := \begin{cases} 1, & k \leqslant k_s \\ \dfrac{s - k_s\Delta}{\Delta}, & k = k_s + 1. \end{cases}$$

2. Define the sequence $x_k \in \mathcal{X}$ for all $k = 1, \ldots, k_s + 1$ as

$$\begin{aligned} x_0 &= x, \\ x_k &= \Phi(\tau_k, x_{k-1}, \theta_k). \end{aligned} \tag{1}$$

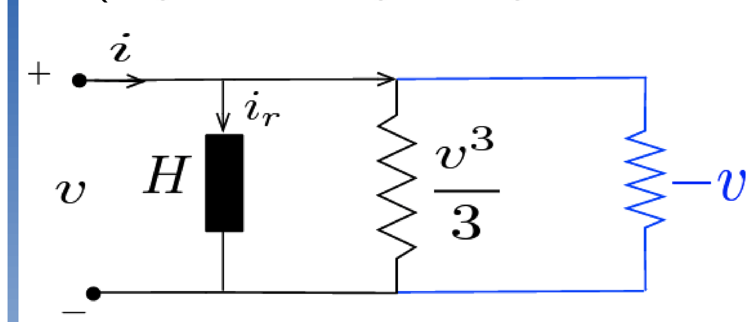
As a consequence of these two steps, we obtain $x_{k_s+1} = \varphi(s, x, u)$.

RNN is the universal approximator of (1)

Experimental evaluation: Predicting excitability in neuronal circuits

Neuronal circuit

(e.g. FitzHugh-Nagumo circuit)

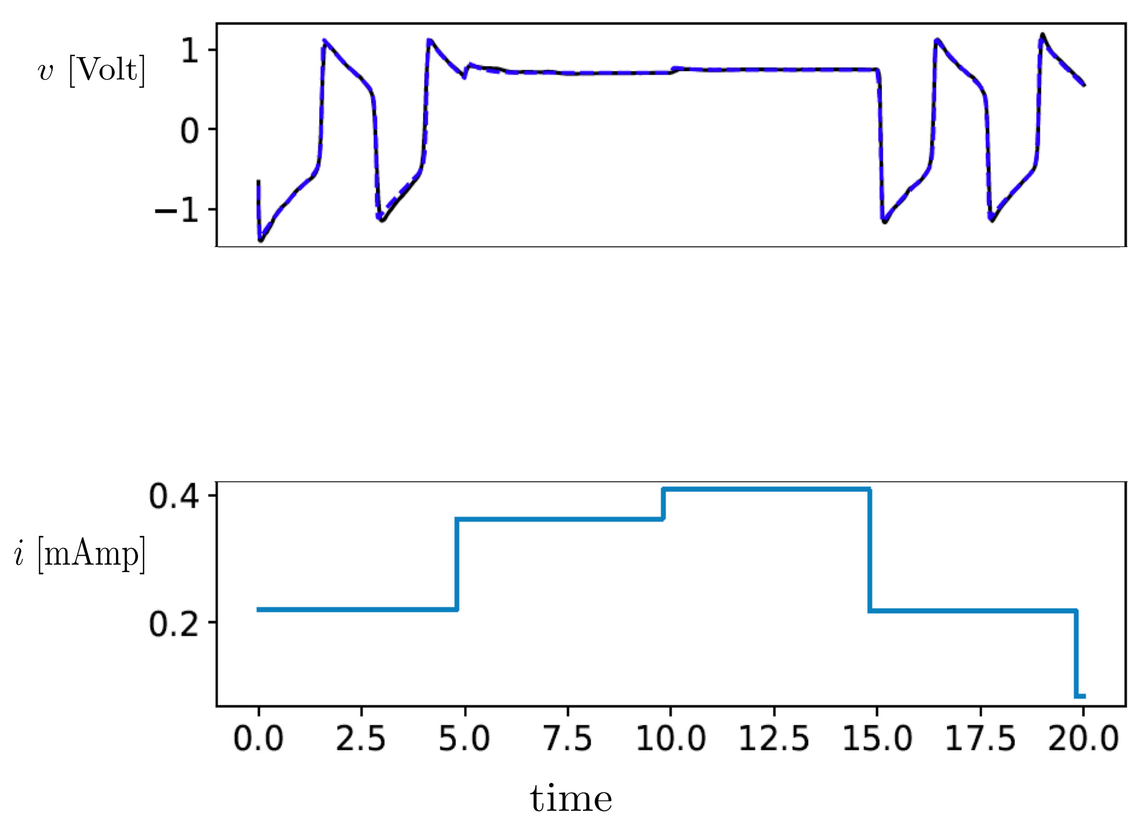


$$H(s) := \frac{LCs^2 + CRs + 1}{Ls + 1}$$

- Parallel interconnection of RLC circuit and tunnel diode.
- Data collected by performing current-clamp experiments on the circuit model.

Predicting excitability

(e.g. FitzHugh-Nagumo circuit)



Generalisation to new input distributions

(e.g. Van der Pol circuit)