Model Approximation of Thermo-Fluidic Diffusion Processes in Spatially Interconnected Structures

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Thermo-Fluidic Processes

Mutual effect of thermal energy on interacting solids and fluids.

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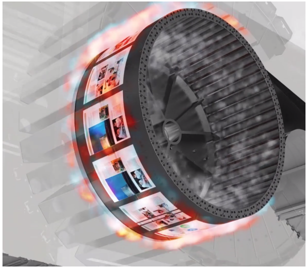
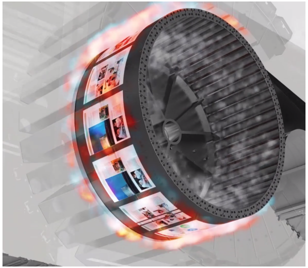
Mutual effect of thermal energy on interacting solids and fluids.

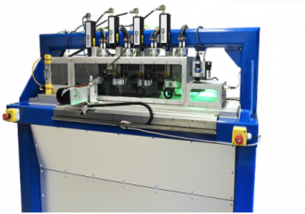
1. Coupled Distributed Parameter Systems.
2. Energy exchange of interacting physical phenomena over boundaries.

Thermo-Fluidic Processes: Examples



Thermo-Fluidic Processes: Examples





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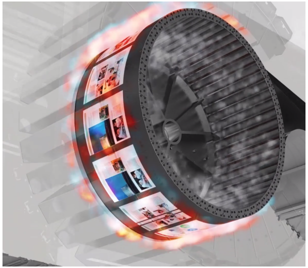
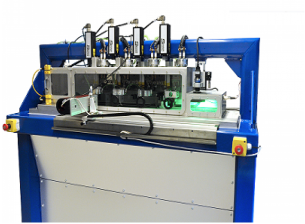
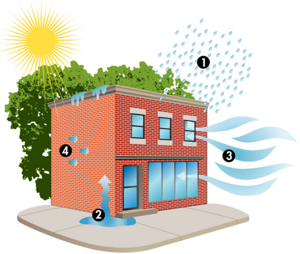
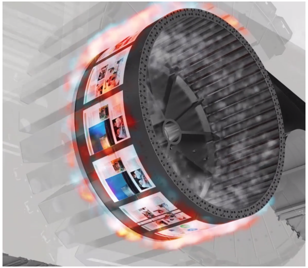
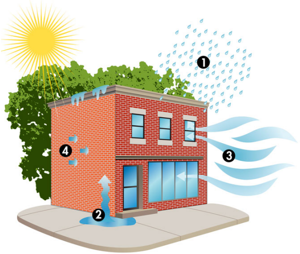


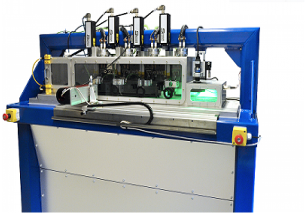


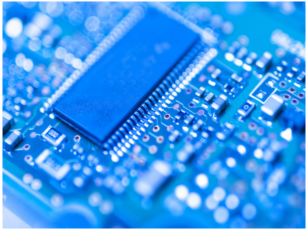


Thermo-Fluidic Processes: Examples









Problem

How to fully exploit the mutual interaction among different physical phenomena for solving model-based problems?

Specifically:

1. Preserving the boundary conditions at the spatial interconnection.
2. Dealing with multi-varibale coupled problems.
3. Solving the boundary control problems.

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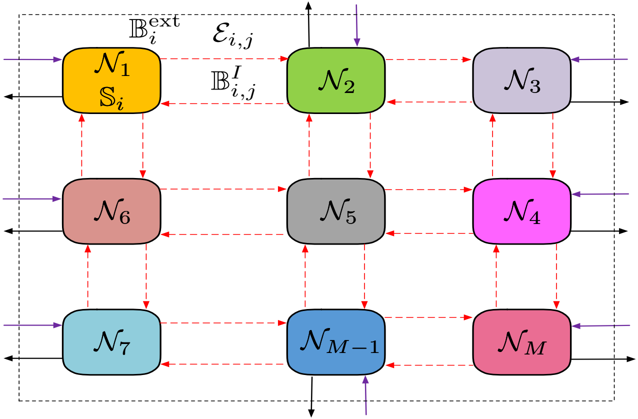
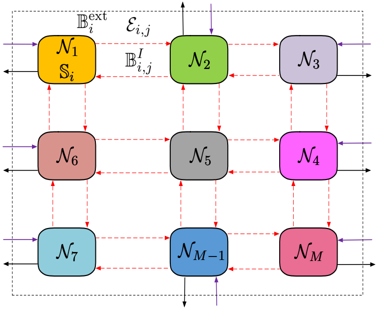
Modeling Framework: Graph Theoretic Description

In a graph, individual systems are a set of Nodes that interact through Edges.



Modeling Framework: Topology Description

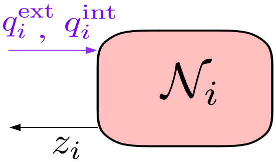


Topology

1. A finite connected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$
2. An adjacency matrix A, with $A = A^T$.

- $\mathcal{N}_i \in \mathcal{N}$ denotes the local dynamics of the node.
- $\mathcal{E}_{i,j} \in \mathcal{E}$ denotes the interconnection of adjacent nodes.

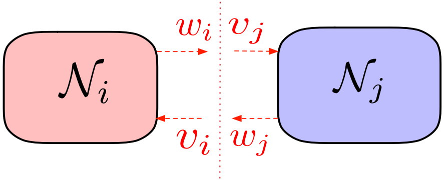
Modeling Framework: Governing Dynamics



$$\text{Model}(\mathrm{D}_i) \begin{cases} \mathcal{E}_i \frac{\partial z_i}{\partial t} = \mathcal{A}_i z_i + \mathcal{B}_i^{\mathrm{int}} q_i^{\mathrm{int}} \\ \mathcal{A}_i z_i := [\nabla \cdot K_i(s) \nabla] z_i \end{cases}$$

$$\text{External boundaries } (\mathrm{B}_i^{\mathrm{ext}}) \begin{cases} \mathcal{H}_i^{\mathrm{ext}} \; z_i = q_i^{\mathrm{ext}}. \\ \mathcal{H}_i^{\mathrm{ext}} \; z_i := [\kappa_i(s) \frac{\partial}{\partial \mathbf{b}_i^{\mathrm{ext}}} + H_i^{\mathrm{ext}}(s)] z_i \end{cases}$$

Modeling Framework: Governing Dynamics



$$\text{Interface boundaries}(\mathrm{B}^{I}_{i,j}) \begin{cases} \begin{bmatrix} w_i \\ w_j \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} v_i \\ v_j \end{bmatrix}, \\ \text{or} \\ \begin{bmatrix} \mathcal{P}^{w}_{i,j} - \mathcal{L}_{i,j}\mathcal{P}^{v}_{i,j} \end{bmatrix} \begin{bmatrix} z_i \\ z_j \end{bmatrix} = 0. \end{cases}$$

Modeling Framework: Summary

1. A finite and connected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$.
2. A symmetric adjacency matrix A.
3. Every node $\mathcal{N}_i$ describes local thermal-fluidic diffusion

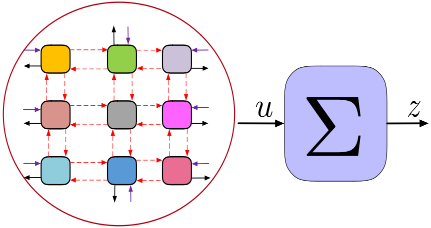
$$\mathcal{N}_i = (\mathbb{S}_i, \mathbb{B}_i^{\text{ext}}, \mathbb{B}_i^{\text{int}}, \mathrm{D}_i, \mathrm{B}_i^{\text{ext}}).$$

4. Every edge $\mathcal{E}_{i,j}$ describes the interconnection of thermal and fluidic process

$$\mathcal{E}_{i,j} = (\mathbb{B}_{i,j}^{I}, \mathrm{B}_{i,j}^{I}).$$

The above defined problem can be proven to be well-posed.

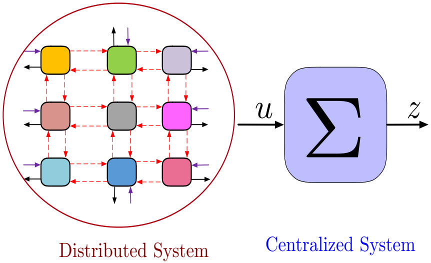
Step 1: Separation of Solution



Distributed System

Centralized System

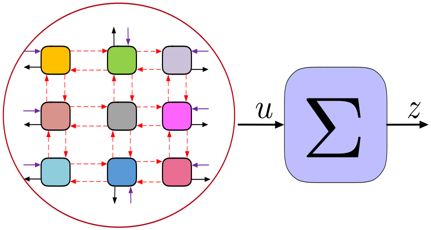
Step 1: Separation of Solution



A) Homogenization

$$\begin{bmatrix} z_1 \\ \vdots \\ z_M \end{bmatrix} := \begin{bmatrix} x_1 \\ \vdots \\ x_M \end{bmatrix} + \mathcal{G} \begin{bmatrix} q_1^{\text{ext}} \\ \vdots \\ q_M^{\text{ext}} \end{bmatrix}$$

Step 1: Separation of Solution



Distributed System Centralized System

B) Define an extended state space (Σ)

$$\underbrace{\begin{bmatrix} \mathcal{E} & 0 \\ 0 & I \end{bmatrix}}_{\mathbf{E}} \begin{bmatrix} \dot{x} \\ \dot{q}^{\text{ext}} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathcal{A} & \mathcal{A}\mathcal{G} \\ 0 & 0 \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x \\ q^{\text{ext}} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathcal{B}^{\text{int}} & -\mathcal{E}\mathcal{G} \\ 0 & I \end{bmatrix}}_{\mathbf{B}} \begin{bmatrix} q^{\text{int}} \\ \dot{q}^{\text{ext}} \end{bmatrix}$$

Step 2. Parametrization & Approximation of Solution

Solution Expansion: $\underbrace{\begin{bmatrix} x \\ q^{\text{ext}} \end{bmatrix}}_{x^e} = \sum_{m=1}^{\infty} \underbrace{\begin{bmatrix} \Theta_m^x(t) & 0 \\ 0 & \Theta_m^q(t) \end{bmatrix}}_{\Theta_m(t)} \underbrace{\begin{bmatrix} \Phi_m^x(s) \\ \Phi_m^q(s) \end{bmatrix}}_{\Phi_m(s)}.$

Signal Projection: $\hat{x}^e(s,t) = \sum_{m=1}^{H} \Theta_m(t)\Phi_m(s)$

Residual System: $\mathbf{R}(x^e) := \mathbf{E}\frac{\partial x^e}{\partial t} - \mathbf{A}x^e - \mathbf{B}u = 0.$

System Projection: $\left\langle \Phi_m, \mathbf{R}(\hat{x}^e) \right\rangle = 0; \quad \forall m \in \{1, \cdots, H\}.$

Can be used for any model based problems.

Feedback Control Problem: LQ optimal problem

The system: $\mathbf{E}\frac{\partial x^e}{\partial t} = \mathbf{A}x^e + \mathbf{B}u.$

Cost Functional: $J(x^e, u) := \int_0^\infty \langle P(s)x^e, x^e\rangle dt + \int_0^\infty \langle R(s)u, u\rangle dt$

Controller: $u = -K(x^e)$

K is the stabilizing, positive semi-definite, self-adjoint solution to the generalized OARE.

Feedback Control Problem: In Finite Dimension

The projected system: $\left\langle \Phi_m, \mathbf{R}(\hat{x}^e) \right\rangle := E_n \dot{\hat{x}}^e = A_n \hat{x}^e + B_n \hat{u}.$

Cost Functional: $J(\hat{x}^e, \hat{u}) := \int_0^\infty \langle \hat{x}^e, P_n \hat{x}^e \rangle dt + \int_0^\infty \langle \hat{u}, R_n \hat{u} \rangle dt$

Controller: $u = -K_n \hat{x}^e$

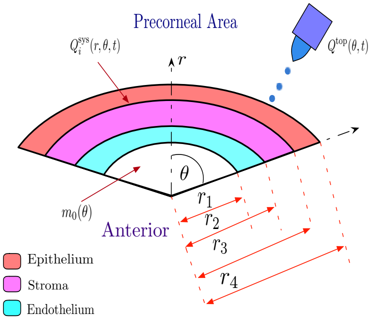
K_n is the stabilizing, positive semi-definite, symmetric solution to the generalized ARE.

Need to solve sparse matrix equalities (Use ADI, Krylov, S).

Example:Boundary Control of Opthalmic Drug Delivery

Goal: Corneal Topography

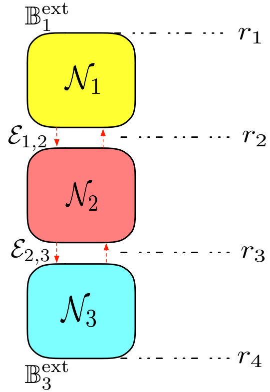
Determine topical drug concentration to achieve dilation across pupil.



Specifics

1. The pre-corneal area has leakage in radial direction due to tears.
2. The anterior chamber has leakage due to drug transport by systemic membrane.
3. Angular directions are insulated.

Example:Boundary Control of Opthalmic Drug Delivery



Topology

1. The nodes are $\{\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3\}$. The edges are $\{\mathcal{E}_{1,2}, \mathcal{E}_{2,3}\}$.
2. $$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$
3. For $\mathcal{N}_i$, $\mathbb{S}_i = [r_i, r_{i+1}]$ for $i = \{1, 2, 3\}$. $\mathbb{B}_1^{\text{ext}} = r_1$, $\mathbb{B}_3^{\text{ext}} = r_4$ and $\mathbb{B}_2^{\text{ext}} = \emptyset$.
4. For every edge $\mathcal{E}_{i,j}$, $\mathbb{B}_{1,2}^{I} = r_2$ and $\mathbb{B}_{2,3}^{I} = r_3$.

Example:Boundary Control of Opthalmic Drug Delivery

Model:

$$\frac{\partial m_i(r,\theta,t)}{\partial t} = \frac{D_i}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial m_i(r,\theta,t)}{\partial r}\Big) + \frac{D_i}{r^2}\Big(\frac{\partial^2 m_i(r,\theta,t)}{\partial \theta^2}\Big).$$

External Boundary conditions:

$$h_1^{-1} D_1 \frac{\partial m_1(r_1,\theta,t)}{\partial r} = [m_1(r_1,\theta,t) - m_0(\theta)],$$

$$h_4^{-1} D_3 \frac{\partial m_3(r_4,\theta,t)}{\partial r} = -[m_3(r_4,\theta,t) - Q^{\text{top}}(\theta,t)].$$

Interface Boundary conditions $i,j = 1,2$**:**

$$\begin{bmatrix} I & -I \\ D_i\frac{\partial}{\partial \mathbf{b}_{i,j}^l} & -D_j\frac{\partial}{\partial \mathbf{b}_{i,j}^l} \end{bmatrix} \begin{bmatrix} m_i(r_{i+1}) \\ m_j(r_{j+1}) \end{bmatrix} = 0.$$

Example:Boundary Control of Opthalmic Drug Delivery

Click!

Conclusion and Future work

Conclusion

1. Generic framework to explicitly include the spatial interconnection.
2. Interconnection is viewed as an exchange of interface input-output.
3. Reduced order solution of a well-posed boundary control systems.

Future Work

1. Interconnection in the view of dissipation.
2. Considering time varying topology (e.g. drying of papers).
3. Including multi-physics models (e.g. fluid dynamics, thermo-elasticity).

Thank You

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