Robust Analysis of Uncertain ODE-PDE Systems Using PIEs

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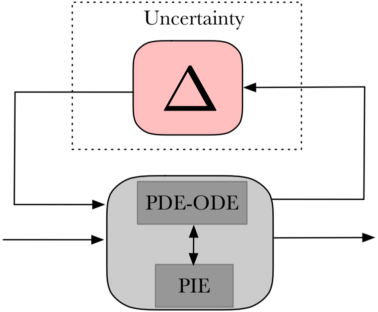
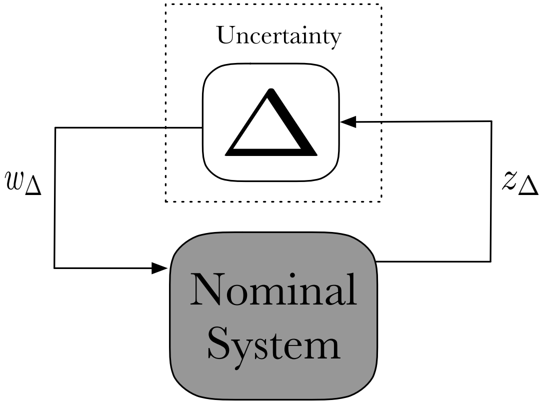




Uncertain ODE Systems

- **Nominal System:**

$$\begin{aligned} \dot{x} &= Ax + Bw_\Delta \\ z_\Delta &= Cx + Dw_\Delta \end{aligned}$$

- **Uncertainty:**

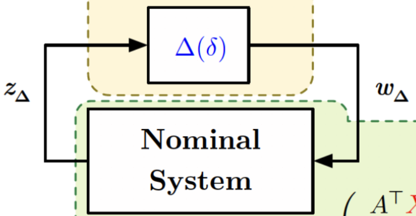
$$\begin{aligned} w_\Delta &= \Delta(\delta, z_\Delta) \\ \Delta &\in \mathbf{\Delta} \end{aligned}$$

$\Delta(\delta, z_\Delta)$ represents uncertainty, parametric variation, static nonlinearities

Revisit: Robust Analysis of Uncertain Systems

Uncertainty is dissipative

$$-\begin{pmatrix} \Delta(\delta)z_\Delta \\ z_\Delta \end{pmatrix}^\top P \begin{pmatrix} \Delta(\delta)z_\Delta \\ z_\Delta \end{pmatrix} \preceq 0$$



Same quadratic supply function!

Nominal system is dissipative

$$\begin{pmatrix} A^\top X + XA & XB \\ B^\top X & 0 \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^\top P \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \prec 0$$

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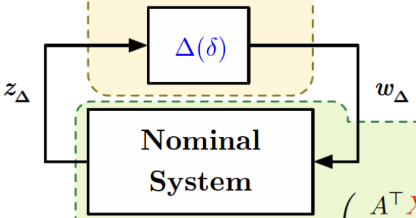

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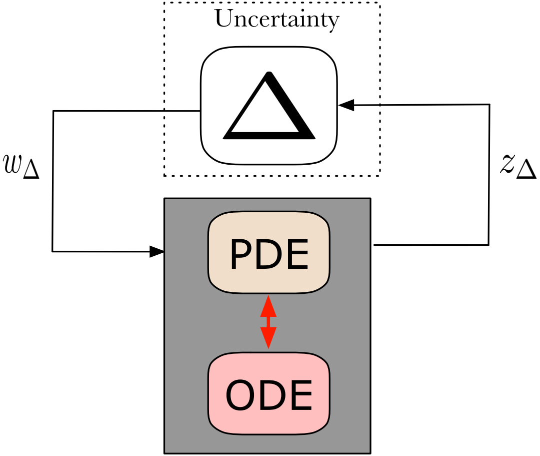
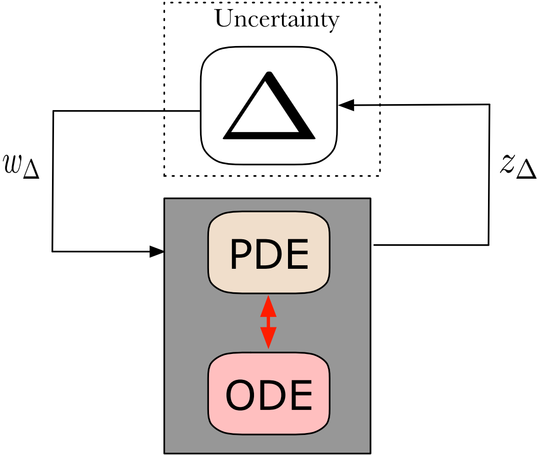
What happens if the nominal system is governed by ODE-PDE model?

Today's Discussion: Robust Analysis of ODE-PDE Systems





3 Ingredients:

1. ODE-PDE systems are equivalent to PIEs
2. Robust analysis of PIEs requires LPIs
3. LPIs are solved using LMIs

LMI tests for uncertain ODE-PDE systems without any discretization!

What is a Partial Integral Equation (PIE)?

PIE is an alternative representation for linear coupled ODE-PDE systems

General form of PIE:

$$\begin{aligned} \mathcal{T}\dot{\mathbf{x}}(t) + \mathcal{T}_u\dot{u}(t) + \mathcal{T}_w\dot{w}(t) &= \mathcal{A}\mathbf{x}(t) + \mathcal{B}_1 w(t) + \mathcal{B}_2 u(t) \qquad \mathbf{x} \in \mathbb{R} \times L_2(a,b) \\ z(t) &= \mathcal{C}_1\mathbf{x}(t) + \mathcal{D}_{11} w(t) + \mathcal{D}_{12} u(t) \\ y(t) &= \mathcal{C}_2\mathbf{x}(t) + \mathcal{D}_{21} w(t) + \mathcal{D}_{22} u(t) \end{aligned}$$

$\mathcal{T}, \mathcal{T}_u, \mathcal{T}_w, \mathcal{A}, \mathcal{B}_i, \mathcal{C}_i, \mathcal{D}_{ij}$ are Partial Integral operators

PIEs are defined by Partial Integral (PI) operators

TU/e

Definition of PI Operator

PI operators are a parametrization of bounded linear operators on $\mathbb{R} \times L_2$.

$$\left(\mathcal{P}\begin{bmatrix} P, & Q_1 \\ Q_2, & \{R_i\} \end{bmatrix}\begin{bmatrix} x_1 \\ \mathbf{x}_2 \end{bmatrix}\right)(s) := \begin{bmatrix} Px_1 + \int_a^b Q_1(s)\mathbf{x}_2(s)ds \\ Q_2(s)x_1 + (\mathcal{P}_{\{R_i\}}\mathbf{x}_2)(s) \end{bmatrix} \tag{4-PI}$$

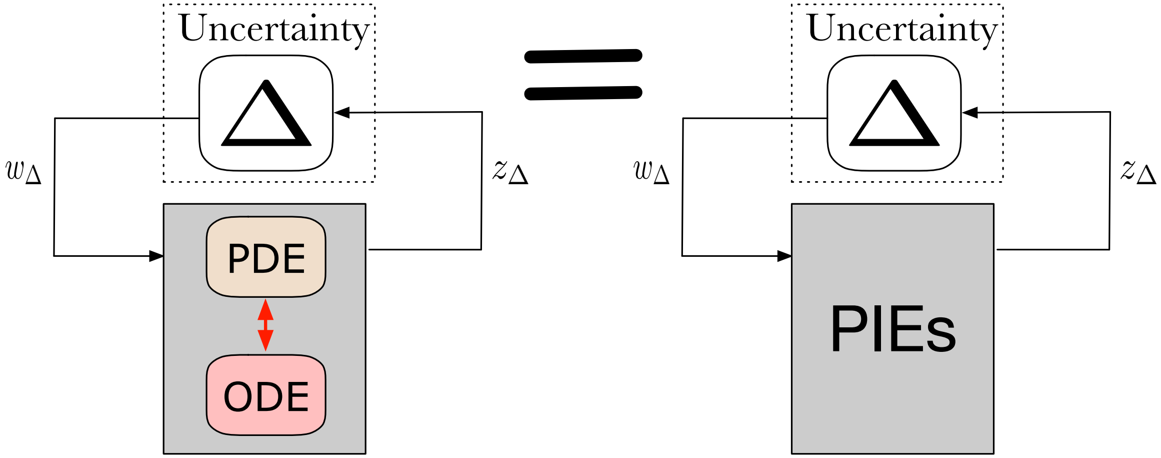
$$(\mathcal{P}_{\{R_i\}}\mathbf{x}_2)(s) := R_0(s)\mathbf{x}_2(s) + \int_a^s R_1(s,\theta)\mathbf{x}_2(\theta)d\theta + \int_s^b R_2(s,\theta)\mathbf{x}_2(\theta)d\theta \tag{3-PI}$$

6 parameters:

- Matrix — P
- Matrix valued polynomials in s — Q_1, Q_2, R_0
- Matrix valued polynomials in s and θ — R_1, R_2

Note: PI operators can be considered as a generalization of matrices on $\mathbb{R} \times L_2$.

Uncertain ODE-PDE Systems Are Equivalent to Uncertain PIEs



Uncertain ODE-PDE Systems Are Equivalent to Uncertain PIEs

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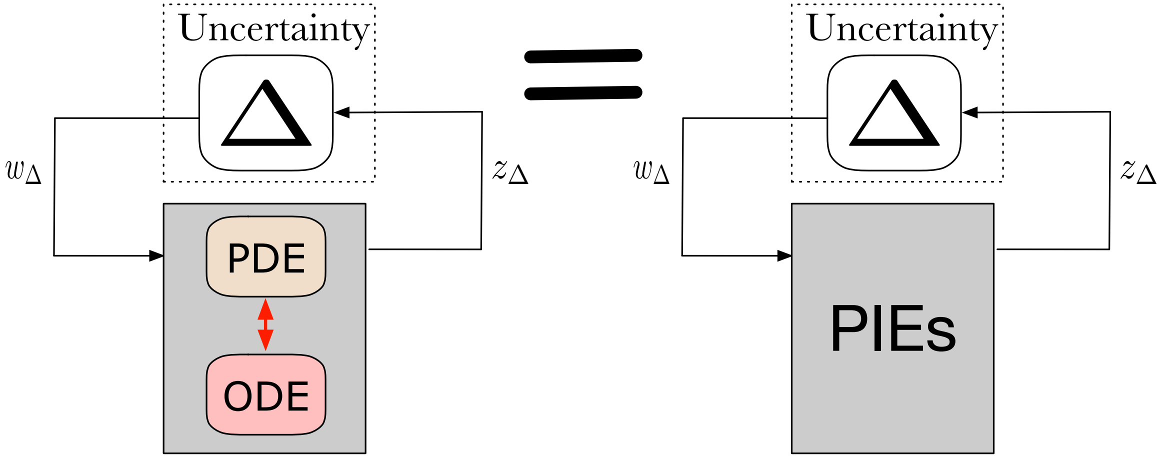
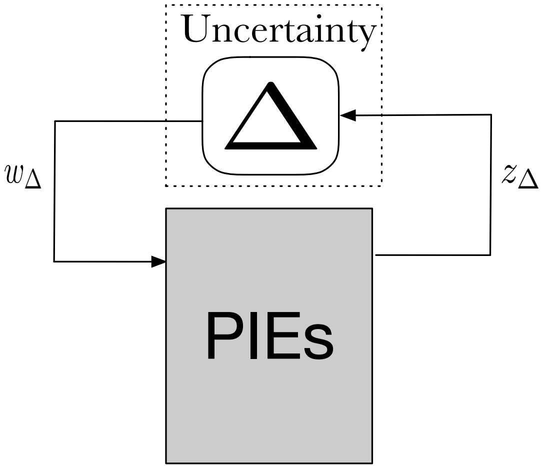


- **Uncertain ODE-PDE** in 1 spatial dimension can be written as a Uncertain PIE
- Solution of uncertain PIE is related to solution of uncertain ODE-PDE via a bijective map



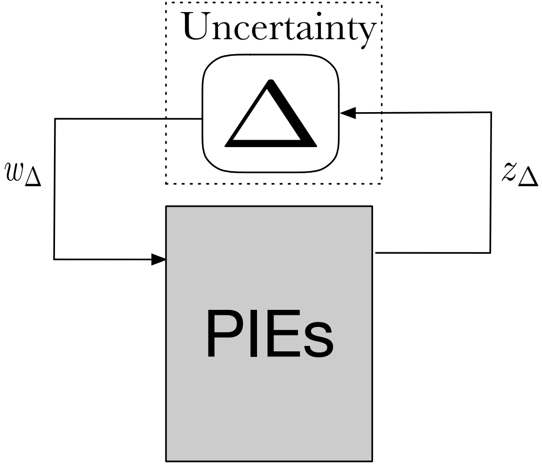
Uncertain PIEs

- **Nominal PIEs:**

$$\begin{aligned}\mathcal{T}\dot{\mathbf{x}} &= \mathcal{A}\mathbf{x} + \mathcal{B}_p w_\Delta \\ z_\Delta &= \mathcal{C}_3 \mathbf{x}\end{aligned}$$

- **Uncertainty:**

$$\begin{aligned}w_\Delta &= \Delta(z_\Delta) \\ \Delta &\in \boldsymbol{\Delta}\end{aligned}$$



Capturing $\Delta \in \boldsymbol{\Delta}$ Using PI Operator

$$\mathbb{M}_{\boldsymbol{\Delta}} := \left\{ \begin{array}{c} \mathcal{M} \mid \forall \Delta \in \boldsymbol{\Delta} \\ \mathcal{M} \text{ is a non-zero PI operator, and} \\ \left\langle \begin{bmatrix} \Delta(q) \\ q \end{bmatrix}, \mathcal{M} \begin{bmatrix} \Delta(q) \\ q \end{bmatrix} \right\rangle \geq 0 \end{array} \right\}.$$

$\mathcal{M} \in \mathbb{M}_{\boldsymbol{\Delta}}$ is a PI multiplier

$$\mathcal{T}\dot{\mathbf{x}} = \mathcal{A}\mathbf{x} + \mathcal{B}_p w_\Delta, \qquad z_\Delta = \mathcal{C}_3\mathbf{x}, \qquad w_\Delta = \Delta(z_\Delta)$$

What we need to test Robust stability for all $\Delta \in \boldsymbol{\Delta}$?

- $\mathcal{M} \in \mathbb{M}_\Delta$ for all $\Delta \in \boldsymbol{\Delta}$
- $\mathcal{P} := \mathcal{P}\begin{bmatrix} P, & Q \\ Q^\top, & \{R_i\} \end{bmatrix}$, $\mathcal{P} = \mathcal{P}^* \succcurlyeq \epsilon I$
- $\begin{bmatrix} 0 & \mathcal{B}_p^* \mathcal{P}\mathcal{T} \\ \mathcal{T}^*\mathcal{P}\mathcal{B}_p & \mathcal{A}^*\mathcal{P}\mathcal{T} + \mathcal{T}^*\mathcal{P}\mathcal{A} + \delta\mathcal{T}^*\mathcal{T} \end{bmatrix} + \begin{bmatrix} I & 0 \\ 0 & \mathcal{C}_3 \end{bmatrix}^* \mathcal{M} \begin{bmatrix} I & 0 \\ 0 & \mathcal{C}_3 \end{bmatrix}^* \preccurlyeq 0$

Testing Robust Stability of Uncertain PIEs

$$\mathcal{T}\dot{\mathbf{x}} = \mathcal{A}\mathbf{x} + \mathcal{B}_p w_\Delta, \qquad z_\Delta = \mathcal{C}_3\mathbf{x}, \qquad w_\Delta = \Delta(z_\Delta)$$

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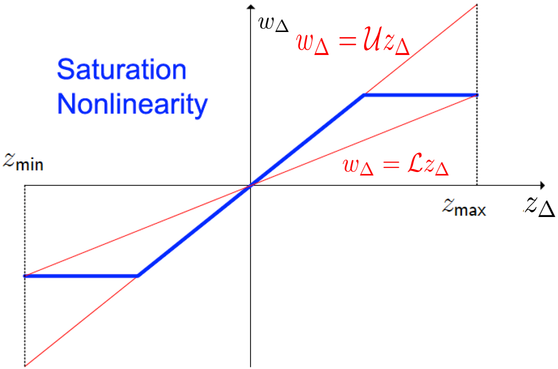
- **A set of inequalities that involve only PI operators a.k.a. Linear PI Inequalities (LPIs)**
- **LPIs require only LMIs to solve them**
- **PIETOOLS can solve LPIs efficiently**

http://control.asu.edu/pietools.html

Example: PI Multiplier for Saturation Nonlinearity



Saturation nonlinearity

- As long as $z_{\min} \leq z_\Delta \leq z_{\min}$,

$$\Big(\Delta(z_\Delta) - \mathcal{U} z_\Delta\Big)^* \Big(\Delta(z_\Delta) - \mathcal{L} z_\Delta\Big) \leq 0$$

- This translate to the following class of PI multipliers

$$\left\{ \tau \begin{bmatrix} I & -\mathcal{U} \\ I & -\mathcal{L} \end{bmatrix}^* \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix} \begin{bmatrix} I & -\mathcal{U} \\ I & -\mathcal{L} \end{bmatrix} \mid \tau \geq 0 \right\} \subset \mathbb{M}_{\boldsymbol{\Delta}}.$$

Diffusion-Reaction Equation with affine non-linearity

$$\frac{\partial v}{\partial t}(s,t) = \lambda v(s,t) + \frac{\partial^2 v}{\partial s^2}(s,t) + f(v(s,t)), \quad (f(v) - v)(f(v) + v) \leq 0$$

For $\lambda > 1.7$, the system is not robustly stable (analytically verifiable)

In summary

We have presented a computational tool to apply LMI-based methods for robust analysis of uncertain ODE-PDE systems

- Coupled ODE-PDEs are represented using PI operators
- Robust analysis is performed by solving LPIs that require LMIs
- PIETOOLS offers a generic and scalable toolbox(plug the model, execute the result)

Perspective

- Same framework in case of determining bounded L_2 gain, synthesizing estimator based controller
- Extension towards dynamic uncertainties: usage of IQCs

Thank You!