Analysis and Control of Networked Infinite Dimensional Systems

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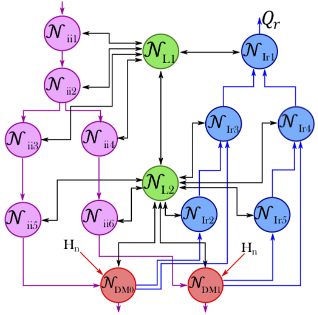
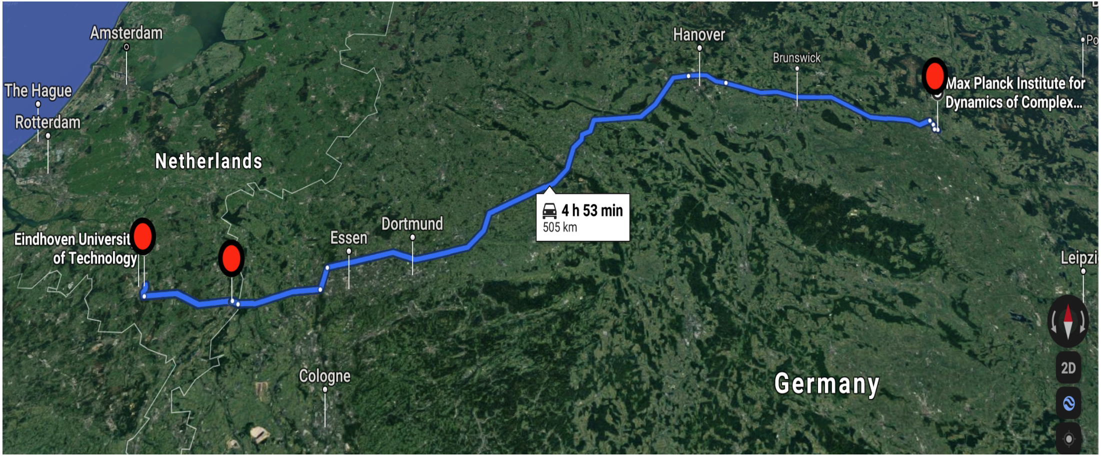


Who Am I?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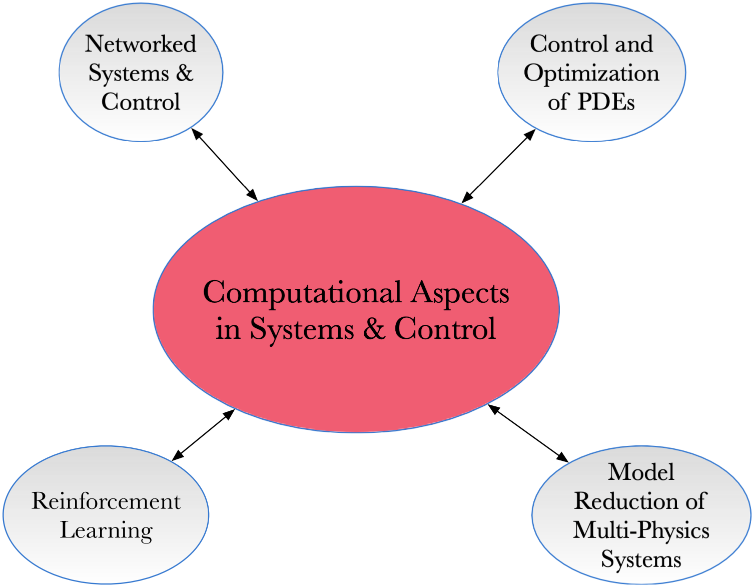
Networked Systems & Control
Control and Optimization of PDEs
Computational Aspects in Systems & Control
Reinforcement Learning
Model Reduction of Multi-Physics Systems

Application: Inkjet Printing

Inkjet Printing

Ink is jetted on the substrate medium in a **predefined** pattern

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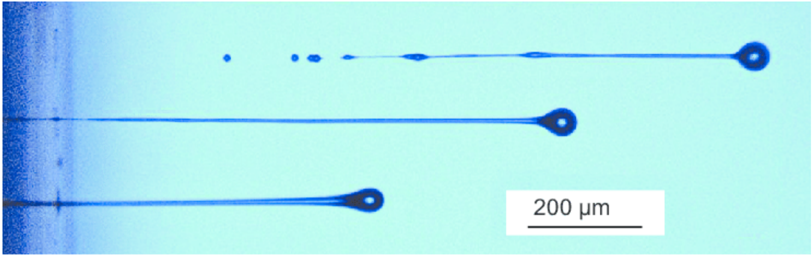
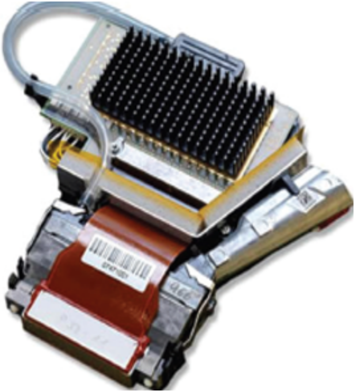
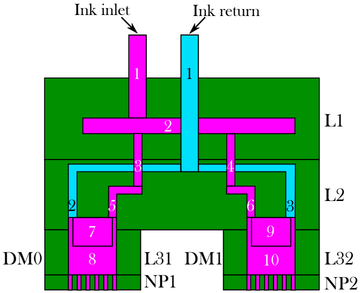


* J. R. Castrejón-Pita; S. J. Willis; Inconsistency in Ink Properties in Printing (Review of Scientific Instruments-2015)

Thermal effects on printhead.





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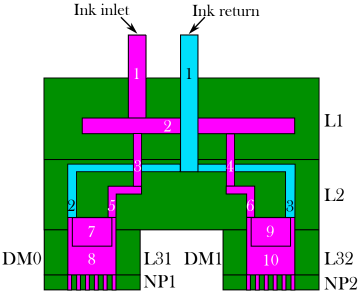
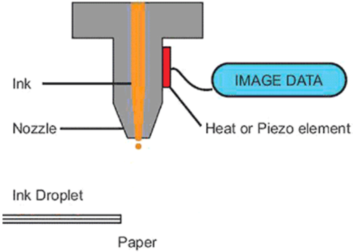
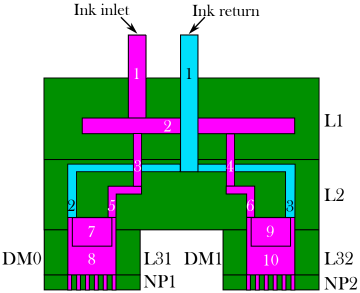


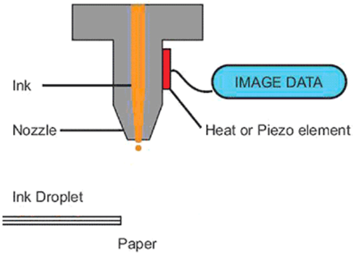


Requirements

- No possibility of additional sensor or actuators
- Good print-quality and high through-put

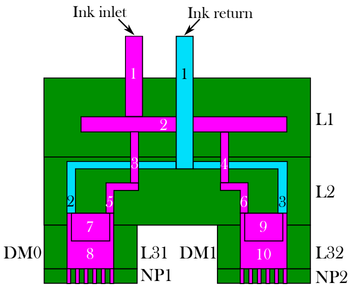
Thermal effects on printhead.





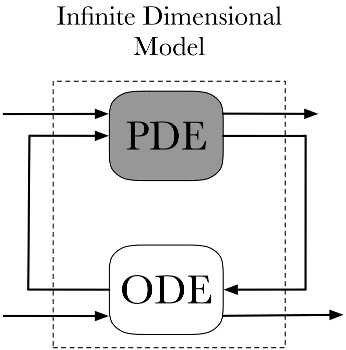
Objective

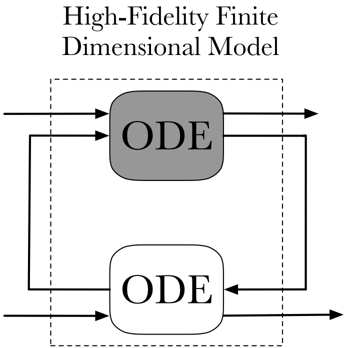
Design a feedback controller that minimizes the gradient of ink-temperature among nozzles without placing additional sensors and actuators?



Common Aspects:

1. Spatial interconnection among multi-physics processes
2. Coupled multi-variable spatio-temporal dynamics (PDEs) and lumped dynamics (ODEs)
3. Energy exchange of interacting physical phenomena over boundaries
4. Requires guaranteed performance in the presence of unmeasured physical quantities

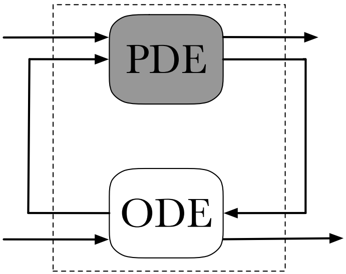




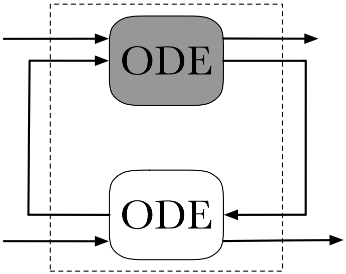
Few aspects of discretization:

1. We have made significant progress due to HPC
2. Curse of dimensionality: moder reduction is a must
3. **Plant** and **model** are not the same

Infinite Dimensional Model



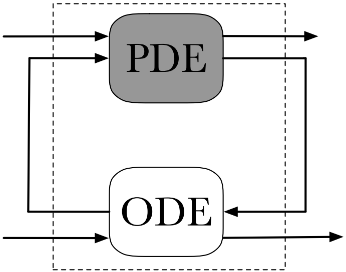
High-Fidelity Finite Dimensional Model



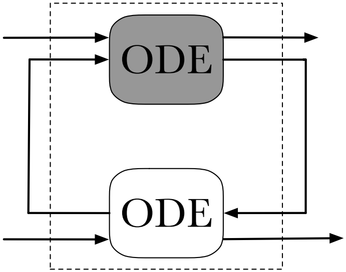
Few aspects of infinite dimensional approach:

1. **Plant** and **model** are the same
2. Methods are problem specific, not scalable
3. Beautiful mathematics, questionable tractability

Infinite Dimensional Model



High-Fidelity Finite Dimensional Model



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How to bridge ideas from finite dimensions to PDEs ?

A new framework for analysis and control of infinite dimensional systems

Specifically

1. Solved using LMIs (polynomial time executable)
2. Generic and scalable (plug the model, execute the result)
3. Does not depend on conventional discretization technique

Why infinite dimensional systems and finite dimensional linear systems are not same?

For linear ODEs with inputs and outputs,

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

A, B, C, D are matrices

e.g. Matrix-valued KYP Lemma (LMI):

$$P \succ 0$$
$$\begin{bmatrix} -\gamma I & D^\top & B^\top P \\ D & I & C \\ PB & C^\top & A^\top P + PA \end{bmatrix} \prec 0.$$

For linear PDEs with inputs and outputs,

$$\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}u(t),$$
$$y(t) = \mathcal{C}\mathbf{x}(t) + \mathcal{D}u(t),$$

$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are differential or unbounded operators

e.g. Operator-valued KYP Lemma (LOI):

$$\mathcal{P} \succ 0$$
$$\begin{bmatrix} -\gamma I & \mathcal{D}^* & \mathcal{B}^*\mathcal{P} \\ \mathcal{D} & I & \mathcal{C} \\ \mathcal{P}\mathcal{B} & \mathcal{C}^* & \mathcal{A}^*\mathcal{P} + \mathcal{P}\mathcal{A} \end{bmatrix} \prec 0$$

Generally, all operators do not inherit properties of a matrix

PI Operators: Integral operators that are parameterized by matrix-valued polynomials

PI operators on $\mathbb{R}^m \times L_2^n[a,b]$

$$\left(\mathcal{P}\begin{bmatrix} P, & Q_1 \\ Q_2, & \{R_{0,1,2}\}\end{bmatrix}\begin{bmatrix} x \\ \mathbf{z}\end{bmatrix}\right)(s) := \begin{bmatrix} Px + \int_a^b Q_1(s)\mathbf{z}(s)ds \\ Q_2(s)x + \underbrace{R_0(s)\mathbf{z}(s)ds} + \underbrace{\int_a^s R_1(s,\eta)\mathbf{z}(\eta)d\eta} + \underbrace{\int_s^b R_2(s,\eta)\mathbf{z}(\eta)d\eta} \end{bmatrix}$$

PI operators are closed under

- Composition. **We denote** $\begin{bmatrix} P, & Q_1 \\ Q_2, & \{R_{0,1,2}\}\end{bmatrix} = \begin{bmatrix} A, & B_1 \\ B_2, & \{C_{0,1,2}\}\end{bmatrix} \times \begin{bmatrix} M, & N_1 \\ N_2, & \{S_{0,1,2}\}\end{bmatrix}$
- Adjoint. **We denote** $\begin{bmatrix} \hat{P}, & \hat{Q}_1 \\ \hat{Q}_2, & \left\{\hat{R}_{0,1,2}\right\}\end{bmatrix} = \begin{bmatrix} P, & Q_1 \\ Q_2, & \{R_{0,1,2}\}\end{bmatrix}^*$
- Addition and Concatenation

Algebraic formula that are computable

Theorem

Let a self adjoint PI operator be defined as

- $\begin{bmatrix} P, & Q \\ Q^\top, & \{R_{0,1,2}\} \end{bmatrix} := \begin{bmatrix} I, & 0 \\ 0, & \{Z_{0,1,2}\} \end{bmatrix}^* \times \begin{bmatrix} P_{11}, & P_{12} \\ P_{12}^\top, & \{Q_{0,1,2}\} \end{bmatrix} \times \begin{bmatrix} I, & 0 \\ 0, & \{Z_{0,1,2}\} \end{bmatrix}, \qquad \{Q_{0,1,2}\} := \{P_{22}, 0, 0\},$
- $\{Z_{0,1,2}\} := \left\{ \begin{bmatrix} \sqrt{g(s)} Z_{d1}(s) \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \sqrt{g(s)} Z_{d2}(s,\theta) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \sqrt{g(s)} Z_{d2}(s,\theta) \end{bmatrix} \right\},$

where $g(s) = (s-a)(b-s)$ *or* $g(s) = 1$ *and* $Z_{d1} : [a,b] \to \mathbb{R}^{d_1 \times n}$, $Z_{d2} : [a,b] \times [a,b] \to \mathbb{R}^{d_2 \times n}$.

Then, the PI operator is positive if and only if the matrix $\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix}$ ***is positive***

Positivity of PI operators can be formulated as positivity of a matrix

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Declaring PI operators

1. `opvar P`: declares a PI operator object
2. `P.P`: A $m \times m$ matrix
3. `P.Q1`, `P.Q2`: A $m \times n$ and a $n \times m$ matrix valued polynomials in s, θ
4. `P.R`: A structure with entities R_0, R_1, and R_2
5. `P.R.R0`: A $n \times n$ matrix valued polynomial in s
6. `P.R.R1`, `P.R.R2` : $n \times n$ matrix valued polynomials in s, θ

Operation on PI operators

```
opvar P1 P2
```

1. Composition: `Pcomp = P1*P2`
2. Adjoint: `Padj = P1'`
3. Addition: `Padd = P1+P2`
4. Concatenation: `Pconc = [P1 P2]` or `Pconc = [P1; P2]`

Example: L_2 induced norm of Volterra integral operators on $L_2[0,1]$

$$\text{minimize } \gamma, \text{ subject to } \mathcal{A}^*\mathcal{A} \le \gamma,$$
$$(\mathcal{A}\mathbf{x})(s) := \int_0^s \mathbf{x}(\theta)d\theta.$$

1. **Declaration of Operator Objects:** Using pvar and opvar

```
» pvar s th gam;
» opvar A; A.R.R1 = 1;
```

2. **Initialization**:

```
» prog = sosprogram([s,th],[gam]);
» prog = sossetobj(prog,gam);
```

3. **Add Constraint:** `» prog = sos_opineq(prog, A'*A-gam);`
4. **Solve the Optimization Problem:**

```
» prog = sossolve(prog);
» Gam = sosgetsol(prog, gam);
```

---

* S. Shivakumar; A. Das; M. Peet; PIETOOLS: A Matlab Toolbox for Manipulation and Optimization of Partial Integral Operators (ACC 2020)

Can we write PDEs in terms of PI Operators?

Consider the following class of PDEs

$$E(s)\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix} = A_0(s)\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix} + A_1(s)\frac{\partial}{\partial s}\begin{bmatrix}\mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix} + A_2(s)\frac{\partial^2}{\partial s^2}\mathbf{x}_3 + B(s)w,$$

$$y = F\mathbf{x}_b + \int_a^b B(s)\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix} \mathrm{d}s + \int_a^b C(s)\frac{\partial}{\partial s}\begin{bmatrix}\mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix} \mathrm{d}s + Dw.$$

Boundary Condition: Sufficient number of boundary conditions: $B_c\mathbf{x}_b = B_w w$

Solution Space: $\mathbf{x} := \mathrm{col}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ belongs to Hilbert or Sobolev space

The conventional notion of states $\mathbf{x} := \text{col}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in L_2 \times H^1 \times H^2$

Question: Are $\mathbf{x}$ independent? **Answer:** No (Fundamental Theorem of Calculus)

$$\mathbf{x}_2(s) = \mathbf{x}_2(a) + \int_a^s \frac{\partial \mathbf{x}_2}{\partial s}(\eta)d\eta = \mathcal{P}\frac{\partial \mathbf{x}_2}{\partial s}$$

$$\frac{\partial \mathbf{x}_3}{\partial s}(s) = \frac{\partial \mathbf{x}_3}{\partial s}(a) + \int_a^s \frac{\partial^2 \mathbf{x}_3}{\partial s^2}(\eta)d\eta = \mathcal{Q}\frac{\partial^2 \mathbf{x}_3}{\partial s^2}$$

$$\mathbf{x}_3(s) = \mathbf{x}_3(a) + s\frac{\partial \mathbf{x}_3}{\partial s}(a) + \int_a^s (s-\eta)\frac{\partial^2 \mathbf{x}_3}{\partial s^2}(\eta)d\eta = \mathcal{R}\frac{\partial^2 \mathbf{x}_3}{\partial s^2}$$

What did we gain?

- $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are PI operators
- Boundary conditions got invoked inside the PI operators

New states: $\mathbf{z} := \textbf{col}\left(\mathbf{x}_1, \frac{\partial \mathbf{x}_2}{\partial s}, \frac{\partial^2 \mathbf{x}_3}{\partial s^2}\right) \in L_2 \times L_2 \times L_2$

The conventional notion of states $\mathbf{x} := \text{col}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) \in L_2 \times H^1 \times H^2$

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$$\mathbf{x}_3(s) = \mathbf{x}_3(a) + s\frac{\partial \mathbf{x}_3}{\partial s}(a) + \int_a^s (s-\eta)\frac{\partial^2 \mathbf{x}_3}{\partial s^2}(\eta) d\eta \qquad = \mathcal{R}\frac{\partial^2 \mathbf{x}_3}{\partial s^2}$$

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- $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are PI operators
- Boundary conditions got invoked inside the PI operators

New states: $\mathbf{z} := \textbf{col}\left(\mathbf{x}_1, \frac{\partial \mathbf{x}_2}{\partial s}, \frac{\partial^2 \mathbf{x}_3}{\partial s^2}\right) \in L_2 \times L_2 \times L_2$

Introducing $\mathbf{z} := \mathrm{col}\left(\mathbf{x}_1, \frac{\partial \mathbf{x}_2}{\partial s}, \frac{\partial^2 \mathbf{x}_3}{\partial s^2}\right)$ as a new state instead of $\mathbf{x} := \mathrm{col}\left(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\right)$

Classical Representation of PDEs:

$$\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}u(t),$$
$$y(t) = \mathcal{C}\mathbf{x}(t) + \mathcal{D}u(t),$$

Boundary Conditions

PI Representation of PDEs:

$$\mathcal{T}\dot{\mathbf{z}}(t) = \mathcal{A}_f\mathbf{z}(t) + \mathcal{B}_f u(t),$$
$$y(t) = \mathcal{C}_f\mathbf{z}(t) + \mathcal{D}_f u(t),$$

Both representations are behaviourally equivalent under the transformation $\mathbf{x} = \mathcal{T}\mathbf{z}$

We have formula for this transformation!

* M. Peet; S. Shivakumar, A. Das; S. Weiland; Discussion Paper: A New Mathematical Framework for Representation and Analysis of Coupled PDEs (CDPS-CPDE, 2019)

For linear ODEs with inputs and outputs,

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

A, B, C, D are matrices

e.g. Matrix-valued KYP Lemma (LMI):

$$P \succ 0$$
$$\begin{bmatrix} -\gamma I & D^\top & B^\top P \\ D & I & C \\ PB & C^\top & A^\top P + PA \end{bmatrix} \prec 0.$$

For linear PI equations with inputs and outputs,

$$\dot{\mathbf{x}}(t) = \mathcal{A}\mathbf{x}(t) + \mathcal{B}u(t),$$
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$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are PI operators

e.g. Operator-valued KYP Lemma (LOI):

$$\mathcal{P} \succ 0$$
$$\begin{bmatrix} -\gamma I & \mathcal{D}^* & \mathcal{B}^*\mathcal{P} \\ \mathcal{D} & I & \mathcal{C} \\ \mathcal{P}\mathcal{B} & \mathcal{C}^* & \mathcal{A}^*\mathcal{P} + \mathcal{P}\mathcal{A} \end{bmatrix} \prec 0$$

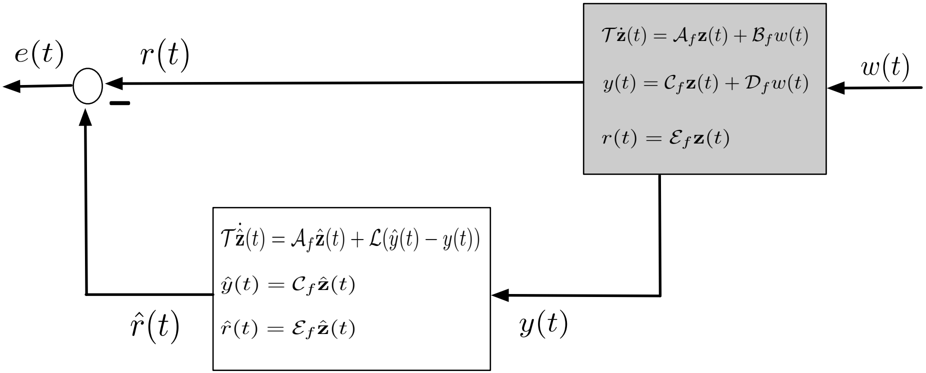
Analogous to matrices, PI operators allow us to solve KYP using LMIs

PI Representation of PDEs:

$$\mathcal{T}\dot{\mathbf{z}}(t) = \mathcal{A}_f\mathbf{z}(t) + \mathcal{B}_f u(t),$$
$$y(t) = \mathcal{C}_f\mathbf{z}(t) + \mathcal{D}_f u(t),$$

Steps to Follow

1. Express the dynamics in terms of PI operators
2. Construct quadratic Lypunov Functions as $V(\mathbf{z}) := \langle \mathbf{z}, \mathcal{T}^*\mathcal{P}\mathcal{T}\mathbf{z}\rangle$
3. Establish the inequalities by taking time-derivative of $V(\mathbf{z})$
4. Enforce operator positivity/ negativity of PI operators
5. Solve LMIs using Semidefinite programming in PIETOOLS



Minimize γ such that $||\hat{r} - r|| \leq \sqrt{\gamma}||w||$

$$\text{min } \gamma, \text{ s.t. } \mathcal{P} \succ 0 \quad \text{with} \quad \mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$$

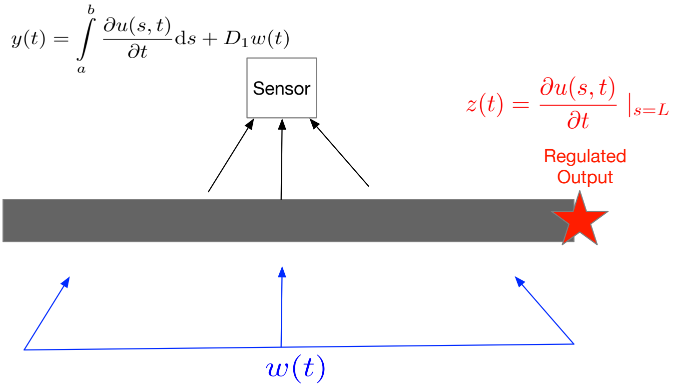
$$\begin{bmatrix} \mathcal{T}^*(\mathcal{P}\mathcal{A}_f + \mathcal{Z}\mathcal{C}_f) + (\mathcal{P}\mathcal{A}_f + \mathcal{Z}\mathcal{C}_f)^*\mathcal{T} & -\mathcal{T}^*(\mathcal{P}\mathcal{B}_f + \mathcal{Z}\mathcal{D}_f) & \mathcal{E}_f^* \\ -(\mathcal{P}\mathcal{B}_f + \mathcal{Z}\mathcal{D}_f)^*\mathcal{T} & -\gamma I & 0 \\ \mathcal{E}_f & 0 & I \end{bmatrix} \prec 0.$$

PDE

$$\frac{\partial^2 u(s,t)}{\partial t^2} = \frac{\partial^2 u(s,t)}{\partial s^2} + B_1(s)w(t)$$

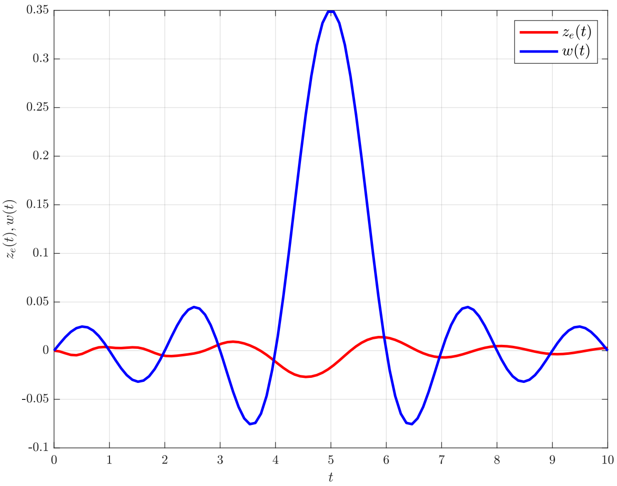
Boundary Conditions

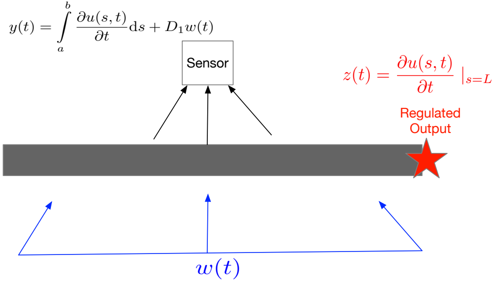
$$u(0,t) = 0,$$
$$\frac{\partial u(s,t)}{\partial s}\Big|_{s=L} = -K\frac{\partial u(s,t)}{\partial t}\Big|_{s=L}$$



Minimize the effect of $w(t)$ on the estimation error $z_e(t) = \hat{z}(t) - z(t)$

Example: $\mathcal{H}_\infty$ Optimal Estimator for Infinite Dimensional System





Back to The Application: Modeling Inkjet Printhead

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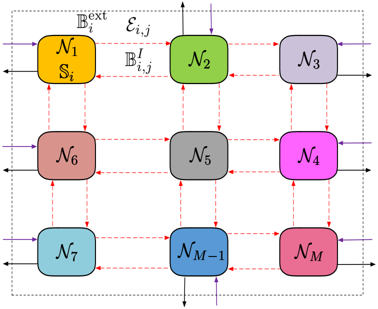


A well-posed dynamic network of infinite dimensional models.

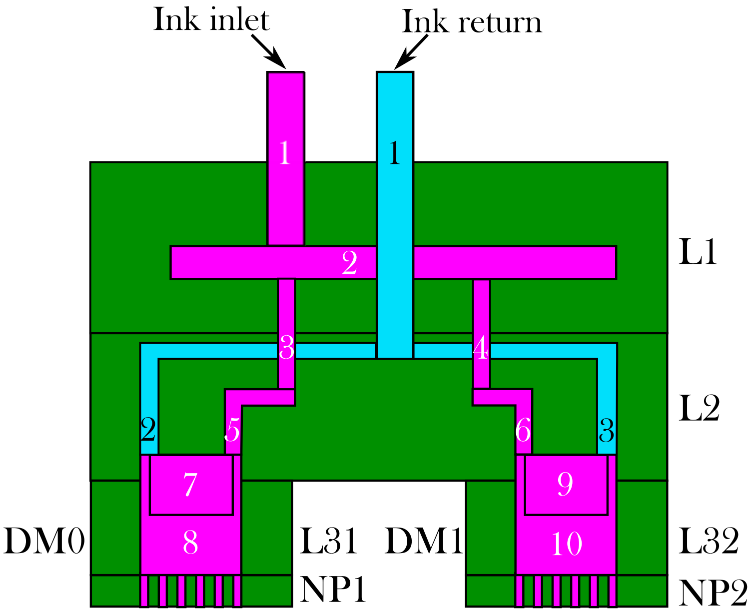
1. A finite and connected graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$.
2. An adjacency matrix A.
3. Every node $\mathcal{N}_i = (\mathcal{S}_i, \mathcal{J}_i, \mathcal{P}_i)$
4. Every edge $\mathcal{E}_{i,j} = (\mathcal{S}_{i,j}^I, \mathcal{M}_{i,j}^I)$.

In Particular:

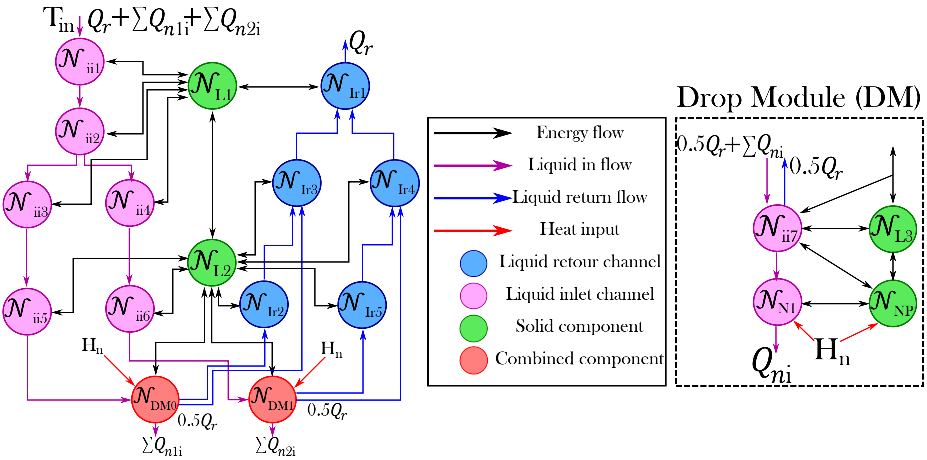
- $\mathcal{N}_i \in \mathcal{N}$ is governed by a set of PDEs or ODEs on a specific domain and under boundary conditions
- $\mathcal{E}_{i,j} \in \mathcal{E}$ denotes the interconnection of adjacent nodes in terms of coupling boundary conditions (for PDEs), algebraic relations (for ODEs)

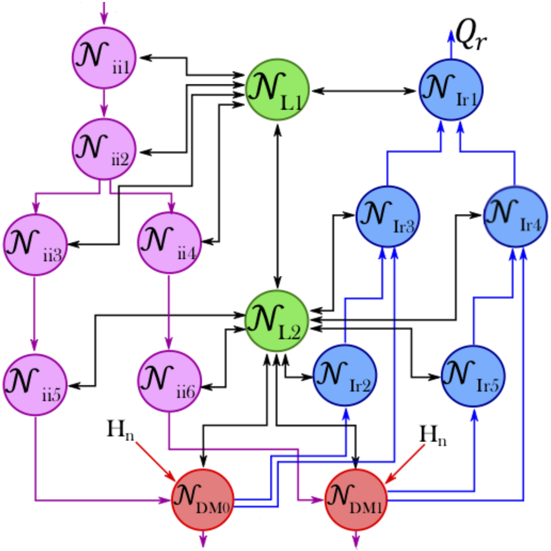
*A. Das; S. Weiland; L. Iapichino; Model Approximation of Spatially Interconnected Thermo-Fluidics (ECC-2018)

Upscaling is Easy in Graph-Theoretic Framework!



Upscaling is Easy in Graph-Theoretic Framework!

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$\mathcal{N}_{ii6}$
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$\mathcal{N}_{L2}$
Q_r
$\mathcal{N}_{Ir1}$
$\mathcal{N}_{Ir2}$
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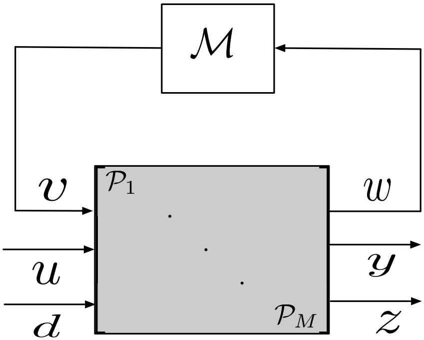
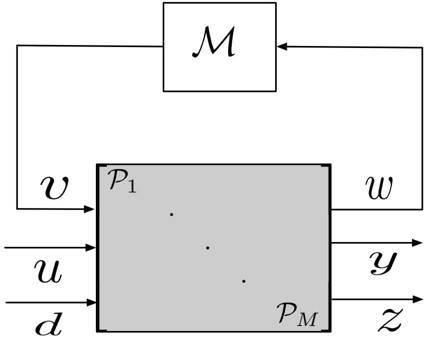
$\mathcal{N}_{DM0}$
$\mathcal{N}_{DM1}$


$\mathcal{M}$
v
w
u
d
y
z
$\mathcal{P}_1$
$\mathcal{P}_M$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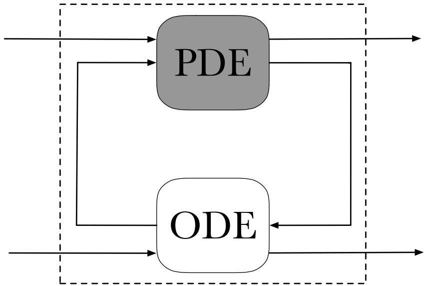
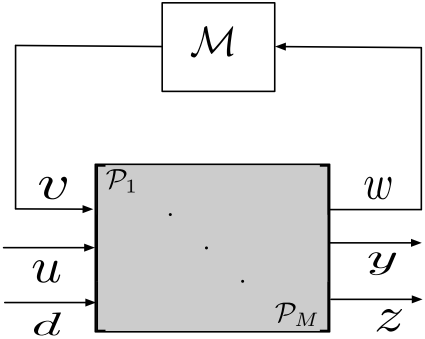
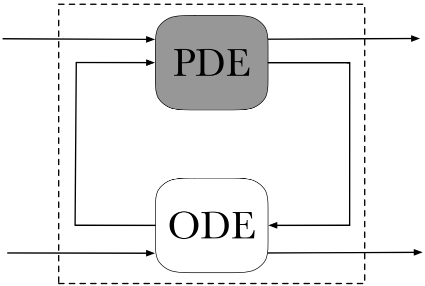

PDE
ODE





Use PI representation of PDE-ODE coupled system

*A. Das; M. Peet; S.Weiland; LMI-Based Synthesis of Coupled PDE-ODE Systems (IEEE TAC-under review)

Tool to apply state-space control theory for PDE-ODE models

Remarks

- A prima-facie verifiable tools for analysis and control of PDE- ODE, time-delay models
- Scalable and polynomial time executable
- Size of the LMI depends on the parametrization of PI operators (upto 20 PDEs in real time)

Future Scope

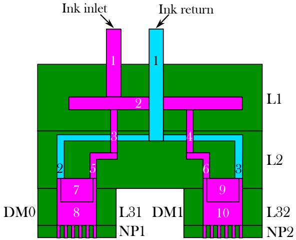
- Extension to higher spatial dimension- **more book-keeping**
- Discretization and MOR of PI - **not yet explored**
- Taking robustness into account (in terms of parametric uncertainty, unmodeled dynamics)- **very little work done**
- Explore distributed control (many controllers under specific communication topology)- **No work available for PDEs**
- Extension to non-linear PDEs (ideas of IQC, multipliers)- **Holy grail**

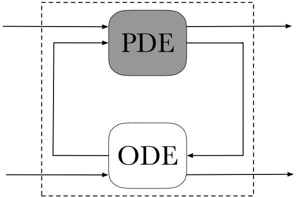
Some Relevant Papers

1. A. Das et. al. (ECC, 2018) : 'Model Approximation of Spatially Interconnected Thermo-Fluidics'
2. M. Peet et. al. (CDPS-CPDE, 2019) : 'Discussion Paper: A New Mathematical Framework for Representation and Analysis of Coupled PDEs'
3. A. Das et. al. (CDC, 2019): '$\mathcal{H}_\infty$ Optimal Estimation of Linear PDE Systems'
4. S. Shivakumar et. al. (CDC, 2019): 'Generalized Input-Output Properties of PDE-ODE Systems'
5. A. Das et. al. (IEEE TAC, under review): 'LMI-Based Synthesis of Coupled PDE-ODE Systems'
6. A. Das et. al. (IEEE TCST, under review): 'Soft Sensor Based In Situ Control of Inkjet Printhead'
7. M. Peet (Automatica, Accepted): 'Modeling Networked and Time-Delay Systems: DDE, DDF, PIEs'

Thank You!

Application: Minimizing Temperature Gradient Among Nozzles





What to do if no additional actuator or sensor is allowed?

Use the already installed piezo-electric element at every individual nozzles

- Use self-sensing capability of piezo-electric elements as soft-sensor at every nozzle
- Use the piezo-electric elements inside non-jetting (idle) nozzles as heating actuators