$\mathcal{H}_\infty$ Optimal Estimation of Linear Coupled PDE Systems

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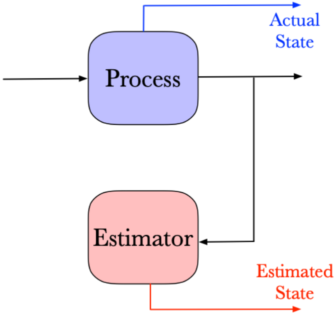
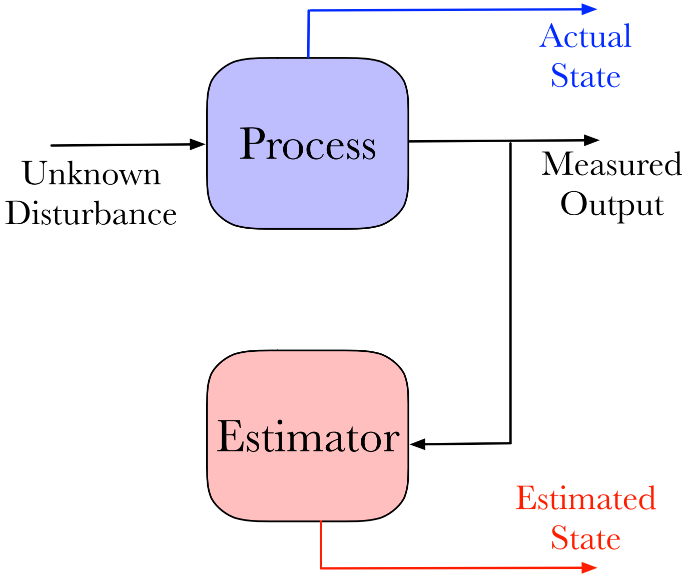
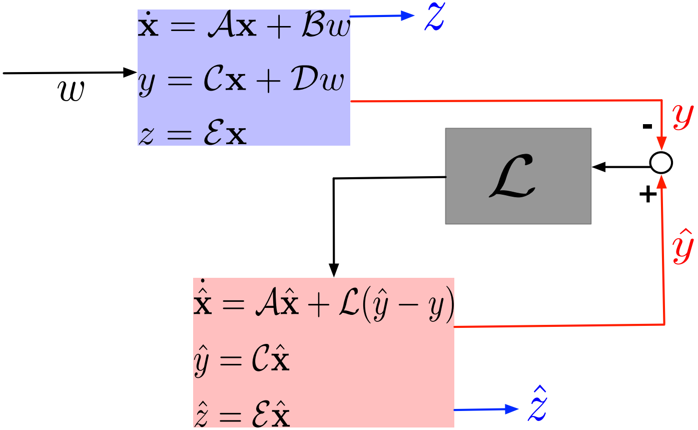




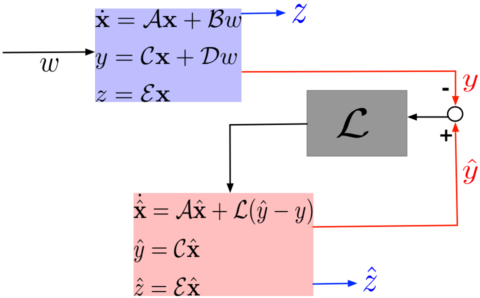
Actual
State
Process
Unknown
Disturbance
Measured
Output
Estimator
Estimated
State



Objective

Determine $\mathcal{L} : \mathcal{Y} \to \mathcal{X}$ and the smallest value of $\rho > 0$, such that

$$\sup_{w \in L_2, w \neq 0} \frac{||\hat{z} - z||_{L_2}}{||w||_{L_2}} < \rho$$



Optimization Problem with LMIs

minimize ρ

$\mathcal{P} \succ 0$

$$\begin{bmatrix} \mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C} + (\mathcal{P}\mathcal{A} + \mathcal{Z}\mathcal{C})^\top & -\mathcal{P}\mathcal{B} - \mathcal{Z}\mathcal{D} & \mathcal{E}^\top \\ -(\mathcal{P}\mathcal{B} + \mathcal{Z}\mathcal{D})^\top & -\rho\mathcal{I} & 0 \\ \mathcal{E} & 0 & -\rho\mathcal{I} \end{bmatrix} \prec 0.$$

$\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$ achieves minimum $\sup_{w \in L_2, w \neq 0} \frac{||\hat{z} - z||_{L_2}}{||w||_{L_2}} < \rho$

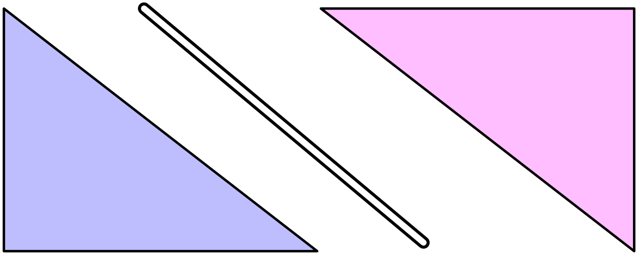
- **PDEs:** For differential/unbounded operators on functions, there is no computational tractability

Develop a new computational framework for synthesizing estimator for PDEs

- **Contribution 1:** Linear PDEs are equivalent to Partial Integral Equations (PIEs)
- **Contribution 2:** Synthesizing estimator for PIEs amounts to solving LMIs
- **Contribution 3:** A scalable toolbox is developed to parse, manipulate and solving PIEs

Construction of PI operators are inspired by matrices

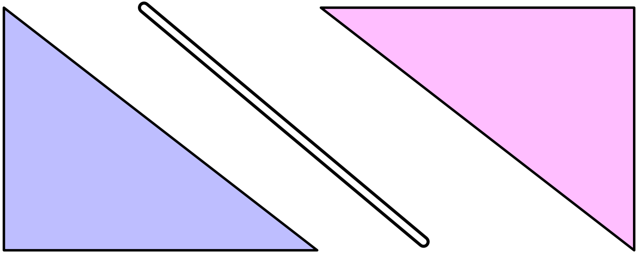
A Matrix on $\mathbb{R}^n$



$$P = \quad R_1 \quad + \quad R_0 \quad + \quad R_2$$

Construction of PI operators are inspired by matrices

3-PI Operators on $L_2^n[a,b]$

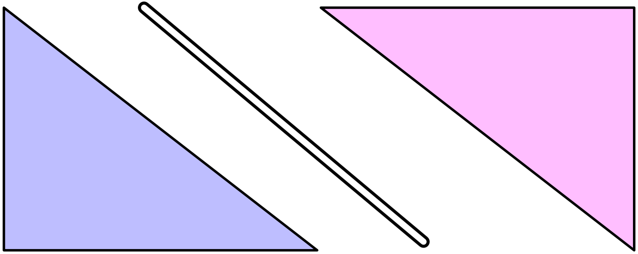


$$P_{\{R_i\}}\mathbf{x}(s) = \int_a^s R_1(s,\theta)\mathbf{x}(\theta)\mathrm{d}\theta + R_0(s)\mathbf{x}(s) + \int_s^b R_2(s,\theta)\mathbf{x}(\theta)\mathrm{d}\theta$$

Partial Integral (PI) Operators

Construction of PI operators are inspired by matrices

3-PI Operators on $L_2^n[a,b]$



$$P_{\{R_i\}}\mathbf{x}(s) = \int_a^s R_1(s,\theta)\mathbf{x}(\theta)\mathrm{d}\theta + R_0(s)\mathbf{x}(s) + \int_s^b R_2(s,\theta)\mathbf{x}(\theta)\mathrm{d}\theta$$

4-PI Operators on $\mathbb{R}^m \times L_2^n[a,b]$

$$\left(\mathcal{P}\begin{bmatrix} P, & Q_1 \\ Q_2, & \{R_i\} \end{bmatrix}\begin{bmatrix} x \\ \mathbf{z} \end{bmatrix}\right)(s) := \begin{bmatrix} Px + \int_a^b Q_1(s)\mathbf{z}(s)ds \\ Q_2(s)x + \mathcal{P}_{\{R_i\}}\mathbf{z}(s) \end{bmatrix}$$

$$P_{\{R_i\}}\mathbf{x}(s) := \int_a^s R_1(s,\theta)\mathbf{x}(\theta)\mathrm{d}\theta + R_0(s)\mathbf{x}(s) + \int_s^b R_2(s,\theta)\mathbf{x}(\theta)\mathrm{d}\theta$$

$$\left(\mathcal{P}\begin{bmatrix} P, & Q_1 \\ Q_2, & \{R_i\}\end{bmatrix}\begin{bmatrix} x \\ \mathbf{z}\end{bmatrix}\right)(s) := \begin{bmatrix} Px + \int_a^b Q_1(s)\mathbf{z}(s)ds \\ Q_2(s)x + \mathcal{P}_{\{R_i\}}\mathbf{z}(s)\end{bmatrix}$$

Declaring PI operators

1. `pvar s th`: declares the independent variables s, θ
2. `opvar P`: declares a PI operator object
3. `P.P`: A $m \times m$ matrix
4. `P.Q1`, `P.Q2`: A $m \times n$ and a $n \times m$ matrix valued polynomials in s, θ
5. `P.R`: A structure with entities R_0, R_1, and R_2
6. `P.R.R0`: A $n \times n$ matrix valued polynomial in s
7. `P.R.R1`, `P.R.R2` : $n \times n$ matrix valued polynomials in s, θ

PIETOOLS: Parsing and Manipulation of PI Operators

TU/e

$$P_{\{R_i\}}\mathbf{x}(s) := \int_a^s R_1(s,\theta)\mathbf{x}(\theta)\mathrm{d}\theta + R_0(s)\mathbf{x}(s) + \int_s^b R_2(s,\theta)\mathbf{x}(\theta)\mathrm{d}\theta$$

$$\left(\mathcal{P}\begin{bmatrix} P, & Q_1 \\ Q_2, & \{R_i\}\end{bmatrix}\begin{bmatrix} x \\ \mathbf{z}\end{bmatrix}\right)(s) := \begin{bmatrix} Px + \int_a^b Q_1(s)\mathbf{z}(s)ds \\ Q_2(s)x + \mathcal{P}_{\{R_i\}}\mathbf{z}(s)\end{bmatrix}$$

PI operators are closed under

- Composition.
  $$\begin{bmatrix} P, & Q_1 \\ Q_2, & \{R_i\}\end{bmatrix} = \begin{bmatrix} A, & B_1 \\ B_2, & \{C_i\}\end{bmatrix} \times \begin{bmatrix} M, & N_1 \\ N_2, & \{S_i\}\end{bmatrix}$$
- Adjoint. $\begin{bmatrix} \hat{P}, & \hat{Q}_1 \\ \hat{Q}_2, & \left\{\hat{R}_i\right\}\end{bmatrix} = \begin{bmatrix} P, & Q_1 \\ Q_2, & \{R_i\}\end{bmatrix}^*$
- Addition and concatenation

Operation on PI operators

`opvar P1 P2`

- Composition: `Pcomp = P1*P2`
- Adjoint: `Padj = P1'`
- Addition: `Padd = P1+P2`
- Concatenation: `Pconc = [P1 P2]` or `Pconc = [P1; P2]`

PIETOOLS has routines to perform these operations

The Positivity of PI Operators as LMIs

TU/e

Theorem

Let a self adjoint 4-PI operator be defined as

- $\begin{bmatrix} P, & Q \\ Q^\top, & \{R_i\} \end{bmatrix} := \begin{bmatrix} I, & 0 \\ 0, & \{Z_i\} \end{bmatrix}^* \times \begin{bmatrix} P_{11}, & P_{12} \\ P_{12}^\top, & \{Q_i\} \end{bmatrix} \times \begin{bmatrix} I, & 0 \\ 0, & \{Z_i\} \end{bmatrix}, \qquad \{Q_i\} := \{P_{22}, 0, 0\},$
- $\{Z_i\} := \left\{ \begin{bmatrix} \sqrt{g(s)} Z_{d1}(s) \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \sqrt{g(s)} Z_{d2}(s,\theta) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \sqrt{g(s)} Z_{d2}(s,\theta) \end{bmatrix} \right\},$

where $g(s) = (s-a)(b-s)$ *or* $g(s) = 1$ *and* $Z_{d1} : [a,b] \to \mathbb{R}^{d_1 \times n}$, $Z_{d2} : [a,b] \times [a,b] \to \mathbb{R}^{d_2 \times n}$.

Then, the 4-PI operator is positive if and only if the matrix $\begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix}$ ***is positive***

By choosing the basis Z_{d1}, Z_{d2}, the positivity of PI operators is equivalent to the positivity of a matrix

Command in PIETOOLS

```
» [prog,P] = sos_posopvar(prog,dim,interval,s,th,deg);
```

Can We Represent PDEs Using PI Operators?

Linear PDEs on $s \in [a, b]$

$$\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{x}_1(s,t)\\ \mathbf{x}_2(s,t)\\ \mathbf{x}_3(s,t)\end{bmatrix} = A_0(s)\begin{bmatrix}\mathbf{x}_1(s,t)\\ \mathbf{x}_2(s,t)\\ \mathbf{x}_3(s,t)\end{bmatrix} + A_1(s)\frac{\partial}{\partial s}\begin{bmatrix}\mathbf{x}_2(s,t)\\ \mathbf{x}_3(s,t)\end{bmatrix} + A_2(s)\frac{\partial^2}{\partial s^2}\mathbf{x}_3(s,t) + B(s)w(t)$$

Boundary Conditions: $B_c x_b(t) = 0$

$$x_b = \text{col}\Big(\mathbf{x}_2(a), \mathbf{x}_2(b), \mathbf{x}_3(a), \mathbf{x}_3(b), \frac{\partial}{\partial s}\mathbf{x}_3(a), \frac{\partial}{\partial s}\mathbf{x}_3(b)\Big), \text{rank}(B_c) = n_2 + 2n_3$$

Solution Space: $\mathbf{x} := \text{col}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ belongs to Hilbert or Sobolev space

Can We Represent PDEs Using PI Operators?

The conventional notion of states $\mathbf{x} := \text{col}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$

Using Fundamental Theorem of Calculus

$$\mathbf{x}_2(s) = \mathbf{x}_2(a) + \int_a^s \frac{\partial \mathbf{x}_2}{\partial s}(\eta) d\eta = \left(\mathcal{P}\frac{\partial \mathbf{x}_2}{\partial s}\right)(s)$$

$$\frac{\partial \mathbf{x}_3}{\partial s}(s) = \frac{\partial \mathbf{x}_3}{\partial s}(a) + \int_a^s \frac{\partial^2 \mathbf{x}_3}{\partial s^2}(\eta) d\eta = \left(\mathcal{Q}\frac{\partial^2 \mathbf{x}_3}{\partial s^2}\right)(s)$$

$$\mathbf{x}_3(s) = \mathbf{x}_3(a) + s\frac{\partial \mathbf{x}_3}{\partial s}(a) + \int_a^s (s-\eta)\frac{\partial^2 \mathbf{x}_3}{\partial s^2}(\eta) d\eta = \left(\mathcal{R}\frac{\partial^2 \mathbf{x}_3}{\partial s^2}\right)(s)$$

What did we gain?

- $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are 3-PI operators
- Boundary conditions got included inside the 3-PI operators

There exists a state transformation $\mathbf{x} = \mathcal{T}\mathbf{x}_f$ where $\mathbf{x}_f := \text{col}\left(\mathbf{x}_1, \frac{\partial \mathbf{x}_2}{\partial s}, \frac{\partial^2 \mathbf{x}_3}{\partial s^2}\right)$ and $\mathcal{T}$ is a 3-PI operator

Can We Represent PDEs Using PI Operators?

The conventional notion of states $\mathbf{x} := \text{col}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$

Using Fundamental Theorem of Calculus

$$\mathbf{x}_2(s) = \mathbf{x}_2(a) + \int_a^s \frac{\partial \mathbf{x}_2}{\partial s}(\eta) d\eta = \left(\mathcal{P} \frac{\partial \mathbf{x}_2}{\partial s}\right)(s)$$

$$\frac{\partial \mathbf{x}_3}{\partial s}(s) = \frac{\partial \mathbf{x}_3}{\partial s}(a) + \int_a^s \frac{\partial^2 \mathbf{x}_3}{\partial s^2}(\eta) d\eta = \left(\mathcal{Q} \frac{\partial^2 \mathbf{x}_3}{\partial s^2}\right)(s)$$

$$\mathbf{x}_3(s) = \mathbf{x}_3(a) + s\frac{\partial \mathbf{x}_3}{\partial s}(a) + \int_a^s (s-\eta) \frac{\partial^2 \mathbf{x}_3}{\partial s^2}(\eta) d\eta = \left(\mathcal{R} \frac{\partial^2 \mathbf{x}_3}{\partial s^2}\right)(s)$$

What did we gain?

- $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ are 3-PI operators
- Boundary conditions got included inside the 3-PI operators

There exists a state transformation $\mathbf{x} = \mathcal{T}\mathbf{x}_f$ where $\mathbf{x}_f := \text{col}\left(\mathbf{x}_1, \frac{\partial \mathbf{x}_2}{\partial s}, \frac{\partial^2 \mathbf{x}_3}{\partial s^2}\right)$ and $\mathcal{T}$ is a 3-PI operator

Can We Represent PDEs Using PI Operators?

Introducing $\mathbf{x}_f := \text{col}\left(\mathbf{x}_1, \frac{\partial \mathbf{x}_2}{\partial s}, \frac{\partial^2 \mathbf{x}_3}{\partial s^2}\right)$ as a new state instead of $\mathbf{x} := \text{col}\left(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\right)$

Partial Differential Equations(PDEs):

$$\frac{\partial}{\partial t}\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix} = A_0\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix} + A_1\frac{\partial}{\partial s}\begin{bmatrix}\mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix} + A_2\frac{\partial^2}{\partial s^2}\mathbf{x}_3 + Bw$$

$$B_c x_b = 0$$

$$y = F x_b + \int_a^b B(s)\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix}\mathrm{d}s + \int_a^b C(s)\frac{\partial}{\partial s}\begin{bmatrix}\mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix}\mathrm{d}s + Dw$$

$$z = G x_b + \int_a^b H(s)\begin{bmatrix}\mathbf{x}_1\\ \mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix}\mathrm{d}s + \int_a^b J(s)\frac{\partial}{\partial s}\begin{bmatrix}\mathbf{x}_2\\ \mathbf{x}_3\end{bmatrix}\mathrm{d}s$$

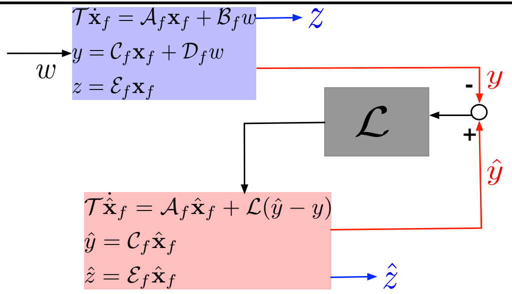
Partial Integral Equations(PIEs):

$$\mathcal{T}\dot{\mathbf{x}}_f = \mathcal{A}_f\mathbf{x}_f + \mathcal{B}_f w$$

$$y = \mathcal{C}_f\mathbf{x}_f + \mathcal{D}_f w$$

$$z = \mathcal{E}_f\mathbf{x}_f$$

Both representations are behaviourally equivalent under the transformation $\mathbf{x} = \mathcal{T}\mathbf{x}_f$



Linear PI Inequalities (LPIs)

minimize ρ

subject to $\mathcal{P} \succ 0 \leftarrow$ **3-PI Operator**

$$\begin{bmatrix} \mathcal{T}^*(\mathcal{P}\mathcal{A}_f + \mathcal{Z}\mathcal{C}_f) + (\mathcal{P}\mathcal{A}_f + \mathcal{Z}\mathcal{C}_f)^*\mathcal{T} & -\mathcal{T}^*(\mathcal{P}\mathcal{B}_f + \mathcal{Z}\mathcal{D}_f) & \mathcal{E}_f^* \\ -(\mathcal{P}\mathcal{B}_f + \mathcal{Z}\mathcal{D}_f)^*\mathcal{T} & -\rho\mathcal{I} & 0 \\ \mathcal{E}_f & 0 & -\rho\mathcal{I} \end{bmatrix} \prec 0 \leftarrow \textbf{4-PI Operator}$$

Then, $\mathcal{L} = \mathcal{P}^{-1}\mathcal{Z}$ achieves minimum value of ρ for which $\sup_{w \in L_2, w \neq 0} \frac{||\hat{z} - z||_{L_2}}{||w||_{L_2}} < \rho$

How To Implement The Estimator in Hardware?

Estimator

$$\mathcal{T}\dot{\hat{\mathbf{x}}}_f(t) = \mathcal{A}_f\hat{\mathbf{x}}_f(t) + \mathcal{P}^{-1}\mathcal{Z}(\mathcal{C}_f\hat{\mathbf{x}}_f(t) - y(t))$$
$$\hat{z}(t) = \mathcal{E}_f\hat{\mathbf{x}}_f(t)$$

It is difficult to derive analytical formula for inversion of 3-PI Operators if $R_1 \neq R_2$

By pre-multiplying with $\mathcal{P}$

$$\mathcal{P}\mathcal{T}\dot{\hat{\mathbf{x}}}_f(t) = \mathcal{P}\mathcal{A}_f\hat{\mathbf{x}}_f(t) + \mathcal{Z}(\mathcal{C}_f\hat{\mathbf{x}}_f(t) - y(t))$$
$$\hat{z}(t) = \mathcal{E}_f\hat{\mathbf{x}}_f(t)$$

Digital Implementation on a gird: Approximate the integration by using numerical discretization

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* In case of $R_1 = R_2$, find the inversion formula in : 'A Convex Solution of the H_∞ Optimal Controller Synthesis Problem for Multi-Delay Systems'- (M. Peet, SICON)

Example: $\mathcal{H}_\infty$ Optimal Estimator for Wave Equation

PDE: Wave Equation in $[0, \mathrm{L}]$

$$\frac{\partial^2 u(s,t)}{\partial t^2} = \frac{\partial^2 u(s,t)}{\partial s^2} + (s^2 - s)w(t)$$

Boundary Conditions

$$u(0,t) = 0,$$
$$\frac{\partial u(s,t)}{\partial s}\,|_{s=L} = -0.5\frac{\partial u(s,t)}{\partial t}\,|_{s=L}$$

Measurement

$$y(t) = \frac{\partial u(s,t)}{\partial s}|_{s=0} + D_1 w(t)$$

Regulated Output

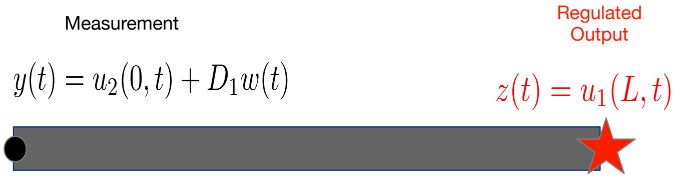
$$z(t) = \frac{\partial u(s,t)}{\partial t}\,|_{s=L}$$

Minimize the effect of $w(t)$ on the estimation error $z_e(t) = \hat{z}(t) - z(t)$

By changing variables

$$u_1 = \frac{\partial u}{\partial t}, u_2 = \frac{\partial u}{\partial s}$$

$$\frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{\partial}{\partial s}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} s^2 - s \\ 0 \end{bmatrix} w$$



Boundary Conditions

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_2(0) \\ u_1(L) \\ u_2(L) \end{bmatrix} = 0$$

Minimize the effect of $w(t)$ on the estimation error $z_e(t) = \hat{z}(t) - z(t)$

1. Using Fundamental Theorem of Calculus

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \int_0^s \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial s} \begin{bmatrix} u_1(\theta) \\ u_2(\theta) \end{bmatrix} \mathrm{d}\theta + \int_s^L \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \frac{\partial}{\partial s} \begin{bmatrix} u_1(\theta) \\ u_2(\theta) \end{bmatrix} \mathrm{d}\theta$$

2. Defining new states $\mathbf{x}_f := \frac{\partial}{\partial s} \begin{bmatrix} u_1(s) \\ u_2(s) \end{bmatrix}$

$$\mathbf{x} = \mathcal{T}\mathbf{x}_f, \mathcal{T} = \left\{ 0, \begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \right\}$$

3. PIE Representation

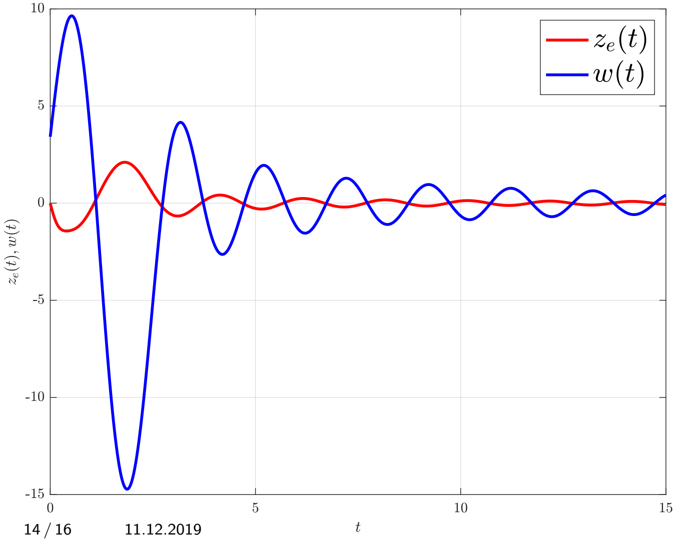
$$\mathcal{T}\dot{\mathbf{x}}_f = \mathcal{A}_f \mathbf{x}_f + \mathcal{B}_f w,$$

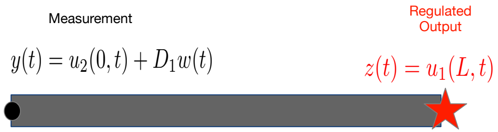
$$\mathcal{A}_f = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, 0, 0 \right\}, \quad \mathcal{B}_f = \left\{ \begin{bmatrix} s^2 - s \\ 0 \end{bmatrix}, 0, 0 \right\}$$

Using PIETOOLS, determine $\mathcal{P}, \mathcal{Z}, \rho$ that achieves minimum $\sup_{w \in L_2, w \neq 0} \frac{||\hat{z} - z||_{L_2}}{||w||_{L_2}} < \rho$

The estimator is implemented on 100 grid points

Figure: $z_e(t) = \hat{z}(t) - z(t)$ with respect to disturbance $w(t)$





Concluding Remarks

In summary

We have presented a computational tool to apply LMI-based methods for Synthesizing $\mathcal{H}_\infty$ optimal estimator for Linear PDEs

- Coupled linear PDEs are represented using PI operators
- $\mathcal{H}_\infty$ optimal estimator is synthesized by solving PI operator inequalities using LMIs
- PIETOOLS offers a generic and scalable toolbox(plug the model, execute the result)

Perspective

- Same framework in case of boundary disturbance
- Easily extendable for PDE-ODE coupled systems, linear time delay systems
- Extension to higher spatial dimension- more book-keeping
- Extendable to take robustness into account (in terms of parametric uncertainty, unmodeled dynamics)

Thank You!