Multi-agent consensus over time-invariant and time-varying signed digraphs via eventual positivity

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Abstract—Laplacian dynamics on signed digraphs have a richer behavior than those on nonnegative digraphs. In particular, for the so-called "repelling" signed Laplacians, the marginal stability property (needed to achieve consensus) is not guaranteed a priori and, even when it holds, it does not automatically lead to consensus, as these signed Laplacians may lose rank even in strongly connected digraphs. Furthermore, in the time-varying case, instability can occur even when switching in a family of systems each of which corresponds to a marginally stable signed Laplacian with the correct corank. In this paper we present novel conditions for achieving consensus on signed digraphs based on the property of eventual positivity, a Perron-Frobenius type of property for signed matrices. The conditions we develop cover both time-invariant and time-varying cases. A particularly simple sufficient condition, valid in both cases, is that the Laplacians are normal matrices. Such condition can be relaxed in several ways. For instance, in the time-invariant case it is enough that the Laplacian has this Perron-Frobenius property on the right side, but not on the left side (i.e., on the transpose). For the time-varying case, convergence to consensus can be guaranteed by the existence of a common Lyapunov function for all the signed Laplacians. All conditions can be easily extended to bipartite consensus.

I. INTRODUCTION

Distributed algorithms for computation and control on networks often rely on a Laplacian-like dynamics to achieve their goal. The underlying assumption that is normally made is that the adjacency matrix of the graph is nonnegative, meaning that the agents collaborate to achieve a common goal. In several applications, however, assuming that the adjacency matrix has nonnegative weights is a limitation. These include social networks, where the individuals can be "friends" or "enemies", computer networks, where computers can trust or mistrust each other, and robot networks, where robots can collaborate or compete to accomplish a task [1], [2], [3], [4], [5], [6],

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[7]. More generally, in all contexts in which antagonism plays a role, it is more appropriate to assume that the weights of the graph can have both positive and negative values, i.e., to consider signed graphs [1], [2]. Other contexts in which signed graphs appear include, e.g., small-disturbance angle stability of microgrids [8], [3].

When a graph is signed, there is more than one way to construct a *signed Laplacian* matrix. In particular, the two main alternatives that have been studied in the literature differ in how the diagonal elements are computed from the signed adjacency matrix. In the terminology of [2], they are denoted "opposing" and "repelling" Laplacian. In this paper we focus on the "repelling" Laplacian, whose main feature is that it always has 0 as an eigenvalue, but it may fail to be stable [9]. Another complication that arises for "repelling" Laplacians is that strong connectivity of the graph no longer guarantees that the Laplacian has corank 1, meaning that even marginally stable "repelling" Laplacians may fail to lead to consensus when their kernel has dimension larger than 1. In the time-invariant case, conditions for stability are provided in [4], [10], [11], [3] for signed undirected graphs, while for signed digraphs some partial results appear in [12], [13], [14]. In the time-varying case, we are not aware of any systematic study (unlike for the "opposing" signed Laplacian, for which an abundant literature exists [5], [6], [7], [15]). The only somewhat related paper we know is [16], which however investigates a different problem, the so-called signed consensus. The interesting aspect that appears when trying to solve the associated time-varying signed "repelling" consensus problem (approximated as a system switching among a certain number of Laplacian matrices) is that stability can be lost even if the time-varying signed Laplacian is marginally stable and with corank 1 for all times. This is in stark contrast to what happens in the "nonnegative" and "opposing" Laplacian cases, and more akin to what happens on "ordinary" (i.e., non Laplacian) time-varying linear systems. In the nonnegative digraph case, in fact, a time-varying Laplacian never diverges, even though it may not converge [17]. Time-varying "opposing" signed Laplacians, on the other hand, converge to zero as soon as one of the switching systems is not structurally balanced [5], [6], [7]. In ordinary linear systems, instead, stability of all matrices of a switching system does not imply stability of the switching system [18], [19], and divergence can occur for certain switching patterns. The analysis that must be performed on our systems is therefore qualitatively different from that required for "nonnegative" and "opposing" time-varying Laplacians (which can never become unstable), as well as from that required for "ordinary" (i.e., not Laplacian) switching linear system (where each matrix is asymptotically stable, while Laplacians are only marginally stable).

The twofold aim of this paper is

1) to provide a thorough stability analysis of signed Laplacians on digraphs in both time-invariant and time-varying cases, and

2) to provide a number of conditions for solving the consensus problem on signed digraphs in both cases.

The conditions we develop are of algebraic nature and rely on Perron-Frobenius (PF) theory. If the canonical way of formulating the PF theorem (for the positive orthant) is to consider a matrix which is nonnegative or Metzler, it has been shown in [20], [21], [22] that the category of matrices enjoying a PF property (namely, spectral radius which is a simple strictly dominating real eigenvalue of the matrix, of positive eigenvector) is strictly larger than nonnegative (or Metzler) matrices, and includes also matrices having some off-diagonal entries that are negative. These matrices are called *Eventually Positive* (EP) when their powers become positive matrices after a certain exponent, and *Eventually Exponentially Positive* (EEP) when the matrix exponential becomes positive after a certain time. EP matrices generalize positive matrices, while EEP matrices generalize Metzler matrices. Eventual positivity has been used by some of us to study consensus-like problems in [12], [13], to represent linear systems which are externally but not internally positive in [23], and to study Laplacian pseudoinverses in [24].

In this paper we show that indeed EEP matrices can provide an effective way to obtain consensus conditions for signed Laplacians. If in the undirected case necessary and sufficient conditions have been obtained in the literature [13], [10], [25], [4], in the directed case (which has so far only been investigated with these tools by us in [12], [13]), only sufficient conditions can be found. The gap between necessity and sufficiency admits a neat interpretation: it corresponds to matrices that obey a "right PF property" but not a "left PF property", i.e., such that their transpose fails to satisfy the PF property. We also show that for weight balanced digraphs our EEP conditions become necessary and sufficient, and that for normal Laplacians they become equivalent to positive semidefinitness of the symmetric part of the Laplacian.

The same notions can be used also in the time-varying case. What is shown in the paper is that a set of matrices that are simultaneously EEP and normal forms a consensus set, i.e., a set such that any switching sequence of matrices from the set (with arbitrary switching times) leads to consensus. A straightforward consequence is that any set of EEP signed Laplacians on undirected graphs is always a consensus set, and convergence to consensus is always guaranteed. That is not true for digraphs in which the signed Laplacians are not normal matrices. In this case, divergence can occur for certain switching patterns, as can be easily shown in examples.

In the paper it is shown that the normality condition is sufficient but not necessary. As a matter of fact, in the time-varying case, normality corresponds to all Laplacians admitting a Common Lyapunov Function (CLF) of quadratic type, in which the matrix associated to the quadratic form is the identity. The class of time-varying Laplacians achieving consensus can be extended considerably if we relax the normality assumption and allow for more general CLFs. In particular, we show how to check the existence of general quadratic CLFs, and also of CLFs which are homogeneous polynomials [26], using Linear Matrix Inequalities (LMI). In order to do so, we need to adapt the methods normally used for families of Hurwitz systems to families of marginally stable systems with the right corank. Unlike e.g. [27], [28], however, the focus is here on consensus, rather than uniform asymptotic stability. For that it is necessary to project the LMIs onto the orthogonal complement of the "agreement subspace", where uniform asymptotic stability tests can then be applied.

The framework of signed graphs finds applications also in discrete-time, e.g., in opinion dynamics, and the algebraic approach we have developed can be applied also to the study of discrete-time signed consensus problems. In fact, when a matrix has negative entries but still has row sums equal to 1, its marginal Schur stability is no longer guaranteed [29]. In this case, the equivalent concept of *Eventually Stochastic* (ES) matrix can be defined and used in an analogue way. In particular, the LMI-based results we obtain in this case complement those obtained for discrete-time switching systems containing mixtures of asymptotically stable and marginally stable modes [30], [27], [31], which normally deal with positive systems only and with uniform asymptotic stability (here we are instead interested in consensus).

Finally, the approach can be generalized also to bipartite consensus, i.e., a form of consensus in which all agents converge to the same value in modulus but not in sign [1]. This type of behavior is often associated to polarization in opinion dynamics, see [1], [5], [6]. Unlike for the "opposing" signed Laplacian, where the concept is tightly linked to the notion of structural balance of the signed digraph [1], here bipartite consensus has to do instead with the signature of the right PF eigenvector, and can occur also when the signed graph is structurally unbalanced.

The rest of this paper is organized as follows: in Section II some preliminary material is presented. The time-invariant case is studied in Section III, and the time-varying case in Section IV. Section V deals with CLF in the time-varying case and Section VI with bipartite consensus. Some open problems are discussed in Section VII.

Part of the material in this paper was presented in the two conference papers [13] and [32]. In particular, a large fraction of the material of Section III appears in [13], and that of Section IV in [32]. The material in Sections V and VI is novel and appears here for the first time.

II. PRELIMINARY MATERIAL

Notations. The real number and integer sets are denoted $\mathbb{R}$ and $\mathbb{Z}$, respectively, while $\mathbb{R}_{\geq}$ and $\mathbb{Z}_{\geq}$ represent respectively the nonnegative real number and nonnegative integer sets. In general, numbers are denoted lowercase letters $x, y, a, b, \ldots$ and lowercase Greek letters $\alpha, \beta, \ldots$ The modulus of a number is denoted $|\cdot|$. For any $x \in \mathbb{R}$, we write $\text{sgn}(x)$ to denote the sign of x and let $\text{sgn}(0) = 0$. Given a positive integer m, let $[m]$ be the set of all positive integers that are no larger than m, i.e., $[m] = \{1, 2, \ldots, m\}$. All vectors are real column vectors denoted with bold lowercase letters $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$ The i-th entry of a vector $\mathbf{x}$ is denoted $[\mathbf{x}]_i$. The Euclidean norm is

denoted $\|\cdot\|$. Matrices are denoted with upper case letters such as $A, B, W, \ldots$ All matrices are real unless stated otherwise. Given a matrix $A \in \mathbb{R}^{n \times n}$, $A^\top$ denotes its transpose and A^k denotes the kth power of A. The (i, j)-th entry of a matrix A is denoted A_{ij} or $[A]_{ij}$; the spectrum of A is denoted $\Lambda(A) = \{\lambda_1(A), \ldots, \lambda_n(A)\}$, while $\mathrm{Re}[\lambda_i(A)]$ indicates the real part of the eigenvalue $\lambda_i(A)$; $\rho(A)$ represents the spectral radius of A, i.e., $\rho(A) = \max_{i \in [n]} |\lambda_i(A)|$, and $\mu(A)$ represents the spectral abscissa of A, i.e., $\mu(A) = \max_{i \in [n]} \mathrm{Re}[\lambda_i(A)]$. For matrices, the symbol ">" (resp. "$\geq$") is to be intended element-wise: $A > 0$ (resp. $A \geq 0$) means $A_{ij} > 0$ (resp. $A_{ij} \geq 0$) for all $i, j = 1, \ldots, n$. Similarly, for vectors, $\mathbf{x} > 0$ (resp. $\mathbf{x} \geq 0$) means $[\mathbf{x}]_i > 0$ (resp. $[\mathbf{x}]_i \geq 0$) for all $i = 1, \ldots, n$. The identity matrix is denoted I_d, with $d > 0$ as the dimension (sometimes omitted). The vectors or matrices with all entries equal to 0 or 1 are all denoted $\mathbf{0}$ or $\mathbf{1}$, with the dimensions depending on the context. Given $A, B \in \mathbb{R}^{n \times n}$, their Kronecker product is denoted $A \otimes B$ and their Kronecker sum $A \oplus B$.

A. Signed graphs and signed Laplacians

A digraph is represented as $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$, where $\mathcal{V} = [n]$ is an index set, an ordered pair $(j, i) \in \mathcal{E}$ denotes a directed link from node j to node i over the set $\mathcal{V}$, and the matrix $A \in \mathbb{R}^{n \times n}$ is the weighted adjacency matrix corresponding to $\mathcal{G}$, with $A_{ij} \neq 0$ if and only if $(j, i) \in \mathcal{E}$. Note that A_{ij} can be positive or negative, which attaches to each edge (j, i) a sign, i.e., $\mathrm{sgn}(A_{ij})$. The graph $\mathcal{G}$ is therefore called a *signed graph*. $\mathcal{G}$ is undirected if $A = A^\top$. $\mathcal{G}$ is called weight balanced if $A\mathbf{1} = A^\top \mathbf{1}$.

Given the signed digraph $\mathcal{G}$ of weighted adjacency matrix A, and denoted $\sigma_i^{\mathrm{in}} = \sum_{j=1}^n A_{ij}$ the weighted in-degree of node i, let us define the Laplacian as

$$L = \Sigma - A \qquad \text{where} \qquad \Sigma = \mathrm{diag}(\sigma_1^{\mathrm{in}}, \ldots, \sigma_n^{\mathrm{in}}). \qquad (1)$$

Since A can have negative entries, σ_i^{in} can even become negative. Notice that the Laplacian (1) is invariant to the presence of digonal elements in A (i.e., of self-loops in $\mathcal{G}$). In our recent paper [2], the Laplacian (1) is referred to as "repelling signed Laplacian", terminology which allows us to distinguish it from a second signed Laplacian (referred to in [2] as "opposing signed Laplacian"), obtained replacing σ_i^{in} with $\sigma_i^{\mathrm{in,abs}} = \sum_{j=1}^n |A_{ij}|$, see [1], [33], [34].

For the graph $\mathcal{G}$, node i is said to be linked to j if there exists an edge sequence $(j, i_1), (i_1, i_2), \ldots, (i_{s-1}, i_s), (i_s, i)$ that is picked from $\mathcal{E}$. We call $\mathcal{G}$ strongly connected if each pair of nodes in $\mathcal{V}$ is linked to each other. A graph $\mathcal{G}$ has a rooted spanning tree if all nodes are linked to j for some $j \in \mathcal{V}$.

B. Matrix theory

Given a square matrix $A \in \mathbb{R}^{n \times n}$, we say that A is Hurwitz stable (resp. Schur stable) if $\mathrm{Re}[\lambda_i(A)] < 0$ (resp. $|\lambda_i(A)| < 1$) for any i, and it is marginally stable (resp. marginally Schur stable) if $\mathrm{Re}[\lambda_i(A)] \leq 0$ (resp. $|\lambda_i(A)| \leq 1$) and $\lambda_i(A)$ such that $\mathrm{Re}[\lambda_i(A)] = 0$ (resp. $|\lambda_i(A)| = 1$) is a simple root of the minimal polynomial of A. A matrix $A \in \mathbb{R}^{n \times n}$ is said to

be irreducible if there does not exist a permutation matrix P such that $P^\top A P$ is block triangular, that is

$$P^\top A P \neq \left[\begin{array}{cc} A_1 & A_2 \\ 0 & A_3 \end{array} \right],$$

where A_1 and A_3 are nontrivial square matrices. The digraph $\mathcal{G}$ of adjacency matrix A is strongly connected if and only if A is irreducible.

$A \in \mathbb{R}^{n \times n}$ is said to be a positive (resp. nonnegative) matrix if $A > 0$, (resp. $A \geq 0$). Given $A \geq 0$, the matrix $B = sI - A$, $s > 0$, is called a Z-matrix (or a negated Metzler matrix). If in addition $s \geq \rho(A)$, then B is called an M-matrix. In particular, an M-matrix B in which $s > \rho(A)$ is nonsingular and such that $-B$ is Hurwitz stable. If instead $s = \rho(A)$, B is a singular M-matrix. If in addition A is irreducible, then $-B$ is also marginally stable.

When $A \in \mathbb{R}^{n \times n}$, the comparison matrix of A, denoted $M(A)$, has $|A_{ii}|$ on the diagonal and $-|A_{ij}|$ in the entry (i, j), $i \neq j$. A matrix A is called an H-matrix if its comparison matrix $M(A)$ is an M-matrix. A is said diagonally dominant (by rows, omitted thereafter) if $|A_{ii}| \geq \sum_{j \neq i} |A_{ij}|$, $i = 1, \ldots, n$.

$A \in \mathbb{R}^{n \times n}$ is said to have corank d if the dimension of the kernel space of A, $\ker(A)$, is d. A is called normal if $AA^\top = A^\top A$. A matrix A is said range symmetric [35] if $\ker(A) = \ker(A^\top)$ (and hence $\mathrm{range}(A) = \mathrm{range}(A^\top)$).

If $A \in \mathbb{R}^{n \times n}$ is symmetric, it is called positive definite (pd) if $\mathbf{x}^\top A \mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq 0$, and positive semidefinite (psd) if $\mathbf{x}^\top A \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$. A pd (resp. psd) matrix A is sometimes indicated $A \succ 0$ (resp. $A \succeq 0$).

C. Perron-Frobenius property and eventual positivity

Definition 1 (Perron-Frobenius property) *A matrix $A \in \mathbb{R}^{n \times n}$ is said to have the (strong) Perron-Frobenius (PF) property, denoted $A \in \mathcal{PF}$, if $\rho(A)$ is a simple real positive eigenvalue of A such that $\rho(A) > |\lambda|$ for all $\lambda \in \Lambda(A)$, $\lambda \neq \rho(A)$, and the corresponding right eigenvector is positive.*

The eigenvalue $\lambda = \rho(A)$ is often referred to as the PF eigenvalue.

Definition 2 (Eventually positive) *A matrix $A \in \mathbb{R}^{n \times n}$ is called Eventually Positive (EP) if there exists $t_0 \in \mathbb{Z}_\geq$ such that A^t is positive for all $t \geq t_0$.*

Following [36], EP matrices will be denoted $A \overset{\vee}{>} 0$. The following necessary and sufficient condition relates EP matrices and PF property.

Theorem 1 *([21], Theorem 2.2) For $A \in \mathbb{R}^{n \times n}$ the following are equivalent:*
  1) *Both $A, A^\top \in \mathcal{PF}$;*
  2) *$A \overset{\vee}{>} 0$;*
  3) *$A^\top \overset{\vee}{>} 0$.*

Also the following lemma will be useful later on.

Lemma 1 *([12], Lemma 1) Consider $A \overset{\vee}{>} 0$ and denote $\mathbf{v} > 0$ its right PF eigenvector. Then any eigenvector $\mathbf{v}_1$ of A such that $\mathbf{v}_1 > 0$ must be a multiple of $\mathbf{v}$.*

Recall that a Metzler matrix (i.e., a negated Z-matrix) is characterized by having a nonnegative exponential: $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \geq 0 \ \forall\, t \geq 0$ [22]. A generalization of exponential nonnegativity is given by the following definition.

Definition 3 (Eventually exponentially positive) *A matrix $A \in \mathbb{R}^{n \times n}$ is called Eventually Exponentially Positive (EEP) if there exists $t_0 \in \mathbb{R}_{\geq}$ such that e^{At} is positive for all $t \geq t_0$.*

The relationship between EP and EEP is provided in the following lemma, which states that EEP matrices are EP matrices with translated eigenvalues.

Lemma 2 *([22], Thm 3.3) A matrix $A \in \mathbb{R}^{n \times n}$ is EEP if and only if $\exists\, d \in \mathbb{R}_{\geq} 0$ such that $A + dI \overset{\vee}{>} 0$.*

EP and EEP matrices generalize respectively positive and Metzler matrices to matrices having some negative off-diagonal entries but still obeying the PF property. In particular, in this paper EP matrices are used to study discrete-time dynamics, and EEP matrices to study continuous-time dynamics. In a way similar to what is done for positive/Metzler matrices, when dealing with EP matrices we consider as dominant eigenvalue the spectral radius (i.e., the PF eigenvalue), while when dealing with EEP matrices the dominant eigenvalue becomes the one with largest real part (i.e., the spectral abscissa of $-L$).

D. Eventually stochastic matrices and signed graphs

Recall that a matrix W is row (resp. column) stochastic if $W\mathbf{1} = \mathbf{1}$ (resp. $\mathbf{1}^\top W = \mathbf{1}^\top$), $0 \leq w_{ij} \leq 1$, and doubly stochastic if it is both row and column stochastic.

Definition 4 (Eventually stochastic) *A matrix $W \in \mathbb{R}^{n \times n}$ is called Eventually Stochastic (ES) if W is EP and $W\mathbf{1} = \mathbf{1}$. If, moreover, $W^\top \mathbf{1} = \mathbf{1}$, W is called Eventually Doubly Stochastic (EDS).*

ES matrices are used to specialize EP matrices to discrete-time dynamics of consensus type (see next Section for the details). Also an ES matrix W can be associated to a signed graph in a natural way by treating W as adjacency matrix of the graph. Positive/negative diagonal elements (i.e., self-loops) are possible for ES matrices, see examples below. In the special case of discrete-time consensus dynamics which is an Euler discretization of a continuous-time system, then the W and L associated to the same signed graph are obviously linked by $W = I - \epsilon L$, where the sampling step size is $\epsilon \in (0, 1/\max_i(L_{ii}))$.

The following lemma follows from the lemma in Sect. VI.C of [12], and from Theorem 1 and Definition 1.

Lemma 3 *If W is ES, then $\rho(W) = 1$ is a simple positive eigenvalue of W such that $\rho(W) > |\lambda|$ for all $\lambda \in \Lambda(W)$, $\lambda \neq \rho(W)$, and the corresponding right and left eigenvectors are positive.*

E. Signed Perron-Frobenius property

Definition 5 (Signed Perron-Frobenius property) *A matrix $A \in \mathbb{R}^{n \times n}$ is said to have the (strong) signed PF property, denoted $A \in \mathcal{SPF}_n$, if $\rho(A)$ is a simple real positive eigenvalue of A such that $\rho(A) > |\lambda|$ for all $\lambda \in \Lambda(A)$, $\lambda \neq \rho(A)$, and the corresponding right eigenvector $\mathbf{v}_r$ is such that $|\mathbf{v}_r| > 0$.*

The special case that interests us is when A, $A^\top \in \mathcal{SPF}_n$ and both $\mathbf{v}_\ell$ and $\mathbf{v}_r$, the left and right eigenvectors associated to $\rho(A)$, have the same sign pattern.

Theorem 2 *(Proposition of Sect. V of [12]) Given $A \in \mathbb{R}^{n \times n}$, then the following are equivalent:*

(i) A, $A^\top \in \mathcal{SPF}_n$, and $\mathbf{v}_\ell$, $\mathbf{v}_r$ such that $[\mathbf{v}_\ell]_i [\mathbf{v}_r]_i > 0$ $\forall\, i = 1, \ldots, n$, or $[\mathbf{v}_\ell]_i [\mathbf{v}_r]_i < 0$ $\forall\, i = 1, \ldots, n$;

(ii) $\exists\, S = \mathrm{diag}(\mathbf{s})$, with $\mathbf{s} = \begin{bmatrix} s_1 & \ldots & s_n \end{bmatrix}^\top$, $s_i = \pm 1$, such that $SAS \overset{\vee}{>} 0$.

The matrix $S = \mathrm{diag}(\mathbf{s})$ in the previous theorem is called a diagonal signature matrix, of signature $\mathbf{s}$. For this special case we shall use the following special definitions.

Definition 6 *A matrix $A \in \mathbb{R}^{n \times n}$ is said*

1) Signed Eventually Positive (SEP) if $\exists$ a diagonal signature matrix S such that SAS is EP;

2) Signed Eventually Exponentially Positive (SEEP) if $\exists$ a diagonal signature matrix S such that SAS is EEP;

3) Signed Eventually Stochastic (SES) if $\exists$ a diagonal signature matrix $S = \mathrm{diag}(\mathbf{s})$ such that SAS is ES and $A\mathbf{s} = \mathbf{s}$.

III. CONSENSUS ON TIME-INVARIANT SIGNED DIGRAPHS

A. Problem formulation

Consider a signed digraph $\mathcal{G}$ over $\mathcal{V}$. The state vector of the agents at time t is given by $\mathbf{x}(t) \in \mathbb{R}^n$, with $[\mathbf{x}(t)]_i$ assigned to agent i for all $i \in \mathcal{V}$. Consider the following two consensus protocols:

1. **Continuous-time protocol**. If L is the signed Laplacian (1) associated to $\mathcal{G}$, the system we consider is

$$\dot{\mathbf{x}} = -L\mathbf{x}. \qquad (2)$$

2. **Discrete-time protocol**. Denote W the signed adjacency matrix associated to $\mathcal{G}$, with the property that $W\mathbf{1} = \mathbf{1}$. The discrete-time protocol is then

$$\mathbf{x}(t+1) = W\mathbf{x}(t). \qquad (3)$$

Definition 7 *We say that the system (2) or (3) achieves consensus if, for all $\mathbf{x}(0) \in \mathbb{R}^n$, there exists $\alpha \in \mathbb{R}$ such that $\mathbf{x}^* = \lim_{t \to \infty} \mathbf{x}(t) = \alpha \mathbf{1}$.*

Problem of interest: find conditions on the signed L and W that guarantee that the systems (2) and (3) achieve consensus.

The presence of signs in L and W complicates things with respect to the unsigned case. For instance, for signed

Laplacians L we have the following easily verifiable properties (similar properties hold also in discrete-time).

Proposition 1 *Consider a signed, strongly connected digraph $\mathcal{G}$. Then for the corresponding Laplacian* (1) *we have*

1) 0 *is always an eigenvalue of right eigenvector* $\mathbf{1}$;
2) *The multiplicity of the 0 eigenvalue can be ≥ 1;*
3) *L need not be diagonally dominant;*
4) *$-L$ need not be marginally stable;*
5) *The quadratic form $x^\top L x$ need not be nonnegative, i.e., $(L + L^\top)/2$ need not be psd.*

The proof is in Appendix A.

As a consequence, L and W can become unstable, and no longer lead to consensus in (2) and (3).

B. Continuous-time case

In this section we first show that for signed graphs a graphical property like connectivity is no longer a characterization of Laplacian rank (Example 1) and hence no longer enough by itself to characterize consensus, while instead an algebraic property like marginal stability (of corank 1) still is (Lemma 4). We then present a general sufficient condition for consensus on digraphs (Theorem 4) based on PF properties, and a necessary and sufficient condition under the more stringent assumption of weight balance (Corollary 1).

From Proposition 1, when $\mathcal{G}$ is a signed graph, L may fail to be diagonally dominant (technically, L need not be an H-matrix), which may or may not lead to loss of stability of $-L$, in a way which is subtle to check, especially since there is no longer a correspondence between irreducibility of the Laplacian and its corank (Property 2 of Proposition 1), as the following example shows.

Example 1 Consider a complete, undirected, signed graph $\mathcal{G}$ whose Laplacian is

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}.$$

It is $\Lambda(L) = \{0, 0, 4, 4\}$ and $\mathbf{1}, \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}^\top$ are both eigenvectors in $\ker(L)$, and 0 has multiplicity 1 in the minimal polynomial of L. Hence L is marginally stable of corank 2. See also [37] for related observations. □

Also a loss of rank can lead to loss of consensus, even in presence of marginal stability. The next lemma highlights the key role of the corank of L, which for signed graphs replaces irreducibility.

Lemma 4 *Given the time-invariant system* (2) *with L a signed Laplacian matrix, consensus is achieved if and only if $-L$ is marginally stable of corank 1.*

The proof is in Appendix A.

Example 1 (cont'd) In this example it is $\lim_{t \to \infty} e^{-Lt} \neq \mathbf{1}\mathbf{c}^\top$, hence the system (2) does not converge to consensus for this corank 2 Laplacian, in spite of marginal stability. □

Our task in the following is therefore to determine conditions that guarantee both the marginal stability and the correct corank of $-L$. We first summarize the known results for undirected graphs, and then develop our new results for digraphs, expanded from their preliminary presentation in [13].

1) Signed undirected graph case: The case of $\mathcal{G}$ signed and undirected has been studied extensively in the literature, mostly in terms of the so-called effective resistance matrix [10], [25], [4]. Following [13] and [3], here we express instead the condition of L psd in terms of EEP matrices.

Theorem 3 ([3], [13]) *Consider an undirected signed graph $\mathcal{G}$ of Laplacian L. L is psd of $\mathrm{corank}(L) = 1$ (and hence $-L$ is marginally stable of $\mathrm{corank}(L) = 1$) iff $-L$ is EEP.*

The proof can be found in [13], Theorem 3.

2) Signed digraph case: In the signed digraph case, the conditions we obtain are no longer necessary and sufficient for marginal stability of $-L$.

Theorem 4 *Consider a signed digraph $\mathcal{G}$, and the corresponding Laplacian L. If $-L$ is EEP, then $-L$ is marginally stable of corank 1, and the system* (2) *converges to*

$$\mathbf{x}^* = \lim_{t \to \infty} \mathbf{x}(t) = \frac{\mathbf{v}_\ell^\top \mathbf{x}(0)\mathbf{1}}{\mathbf{v}_\ell^\top \mathbf{1}},$$

where $\mathbf{v}_\ell$ is the left eigenvector of L relative to 0. Viceversa, if $-L$ is marginally stable and of corank 1 then $\exists$ a scalar $d \geq 0$ such that $dI - L \in \mathcal{PF}$.

The proof is in Appendix A.

Remark 1 The gap between the two conditions of Theorem 4 corresponds to matrices L s.t. $B = dI - L \in \mathcal{PF}$ for some $d \geq 0$, but $B^\top \notin \mathcal{PF}$ for all $d \geq 0$. The stability of such class of Laplacians cannot be determined a priori, as the following two examples show.

Example 2 In correspondence of

$$L = \begin{bmatrix} -0.4 & 0.7 & 0 & -0.3 \\ -1.4 & 1.6 & 0.2 & -0.4 \\ -0.7 & 0 & 2.8 & -2.1 \\ 0 & 0 & -1.3 & 1.3 \end{bmatrix}$$

it is $\Lambda(L) = \{0, 0.7325 \pm 0.1220i, 3.8349\}$, i.e., $-L$ is marginally stable, even though the left eigenvector associated to 0 is not positive. For $d > 1.92$, $B = dI - L \in \mathcal{PF}$ but $B^\top \notin \mathcal{PF}$. Notice also that L has both positive and negative values on the diagonal. □

Example 3 For

$$L = \begin{bmatrix} 1.4 & -1.9 & 0.5 & 0 \\ 0 & 1 & 0 & -1 \\ 0.3 & -0.2 & 0.1 & -0.2 \\ -1.8 & 0 & 0 & 1.8 \end{bmatrix}$$

it is $\Lambda(L) = \{-0.0890, 0, 2.1945 \pm 1.2509i\}$, i.e., $-L$ is unstable. Clearly $x^\top L x$ is not nonnegative. Also in this case the left eigenvector associated to 0 is not positive, and, for $d > 2.6$, $B = dI - L \in \mathcal{PF}$ but $B^\top \notin \mathcal{PF}$. $\square$

The following is instead an example of EEP matrix.

Example 4 For

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2.6 & -2.6 & 0 \\ -0.3 & 0 & 1.4 & -1.1 \\ -0.9 & 0.2 & -0.9 & 1.6 \end{bmatrix}$$

we have $\Lambda(L) = \{0, 1.6956 \pm 0.9452i, 3.2089\}$. For $d \geq 1.61$ it is $B = dI - L \overset{\vee}{>} 0$. $\square$

In the case of L weight balanced, however, there is no gap between necessity and sufficiency.

Corollary 1 *Consider a signed digraph $\mathcal{G}$ such that the corresponding Laplacian* (1) *is weight balanced. Then the following conditions are equivalent:*
(i) $-L$ is EEP;
(ii) $-L$ is marginally stable of corank 1;
Furthermore, if L is normal then (i) and (ii) are equivalent to
(iii) $L_s = (L + L^\top)/2$ is psd of corank 1.

The proof is in Appendix A.

Remark 2 Corollary 2 of [13] erroneously claims that the equivalence (ii) $\iff$ (iii) in Corollary 1 is valid in the more general case of weight balance L, which is not true. Only one direction is valid ((iii) $\implies$ (ii)), as shown in Corollary 2. As for the other direction, a complication arises for instance from the fact that for L weight balanced but not normal L_s may acquire negative diagonal elements even if $-L$ is marginally stable. L_s with negative diagonal elements obviously cannot be psd. However, even when L_s has positive diagonal, it is not guaranteed to be psd, see Example 5. Normality of L guarantees instead psd of its symmetric part, although it is not necessary, see Example 6.

Corollary 2 *If L_s is psd of corank 1, then L is weight balanced and marginally stable of corank 1.*

The proof is in Appendix A.

Example 5 For

$$L = \begin{bmatrix} 0.23 & 0 & -0.28 & 0.05 \\ -0.01 & 0.03 & 0.02 & -0.04 \\ 0.05 & -0.03 & 0.04 & -0.06 \\ -0.27 & 0 & 0.22 & 0.05 \end{bmatrix}$$

it is $\Lambda(L) = \{0, 0.1443 \pm 0.1859i, 0.0514\}$, i.e., $-L$ is marginally stable of corank 1. Moreover, $L\mathbf{1} = L^\top \mathbf{1} = 0$ and, for $d > 0.1919$, $B = dI - L \overset{\vee}{>} 0$. However, $\Lambda(L_s) = \{-0.0446, 0, 0.0404, 0.3441\}$, i.e., L_s is not psd. $\square$

Example 6 For

$$L = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & 1 & 0 & 0 \\ -1 & -1 & 2 & 0 \\ 1 & -1 & -1 & 1 \end{bmatrix},$$

which is not normal, it is $\Lambda(L) = \{0, 1.5 \pm 1.323i, 2\}$, i.e., $-L$ is marginally stable of corank 1, and $\Lambda(L_s) = \{0, 0.7192, 1.5, 2.7808\}$, i.e., L_s is psd of corank 1. $\square$

Observe that Theorem 4 and Corollary 1 do not explicitly assume that $\mathcal{G}$ is strongly connected. As already mentioned, it follows from Property 2 in Proposition 1 that for signed graphs strong connectivity of $\mathcal{G}$ (and irreducibility of L) does not always lead to $\text{corank}(L) = 1$, see Example 1. Similarly, $\text{corank}(L) = 1$ by itself need not imply L irreducible, even though it implies the existence of a rooted spanning tree. Irreducibility follows if in addition L is weight balanced. In particular, irreducibility of L (and strong connectivity of $\mathcal{G}$) are implied by any of the conditions (i) or (ii) of Corollary 1. These properties are formalized in the following lemma.

Lemma 5 *Let $\mathcal{G}$ be a signed digraph with Laplacian L.*
1) If L is of corank 1, then $\mathcal{G}$ has a rooted spanning tree.
2) If $-L$ is EEP or if L is weight balanced and of corank 1, then L is irreducible (and $\mathcal{G}$ is strongly connected).

The proof is in Appendix A.

C. Discrete-time protocol

In this section we use the notion of ES to establish equivalent results for the discrete-time system (3) on a signed graph $\mathcal{G}$. Since neither the undirected graph case nor the digraph case have been analyzed in the literature so far, in this section we discuss both in detail. In particular, we provide a necessary and sufficient condition for consensus on undirected graphs (Theorem 5), while for digraphs we provide a general sufficient condition (Theorem 6), and a necessary and sufficient condition for the special case of weight balance (Corollary 3). We start by observing that also in this case the graphical point of view by itself no longer suffices: irreducibility of W does not guarantee simplicity of the dominant eigenvalue, which in turn may hamper consensus even though W is still marginally Schur stable.

Example 7 Consider $W = I - L/10$, where L is given in Example 1. By construction $W\mathbf{1} = W^\top \mathbf{1} = \mathbf{1}$ and W irreducible. Nevertheless, $\Lambda(W) = \{0.6, 0.6, 1, 1\}$, i.e., the dominant eigenvalue has multiplicity 2 and W is marginally Schur stable. $\square$

Irreducibility of W follows, e.g., when we have the ES property.

Lemma 6 *Consider a signed digraph $\mathcal{G}$ of adjacency matrix W such that $W\mathbf{1} = \mathbf{1}$. If W is ES or if W is weight balanced with $\lambda(W) = 1$ which is simple and strictly dominant, then W is irreducible (and $\mathcal{G}$ strongly connected).*

Since W ES implies W EP, the proof resembles that of Lemma 5 and is therefore omitted.

We now move to consensus. The following lemma states that consensus is equivalent to marginal Schur stability of W plus simplicity and strict dominance of the $\lambda(W) = 1$ eigenvalue.

Lemma 7 *Consider the system* (3) *and let the signed weighted adjacency matrix W satisfy $W\mathbf{1} = \mathbf{1}$. Consensus is achieved if and only if W is marginally Schur stable with the eigenvalue $\lambda(W) = 1$ which is simple and strictly dominant.*

The proof is in Appendix A.

Example 8 Consider $W = I - L/2$, where L is given in Example 1. By construction $W\mathbf{1} = W^\top\mathbf{1} = \mathbf{1}$. $\Lambda(W) = \{-1, -1, 1, 1\}$, i.e., $\rho(W) = |\lambda_i(W)|$ for all i. W is marginally stable, but simplicity and strict dominance are missing. Since $\lim_{t\to\infty} W^t$ does not exist, the system (3) does not converge to consensus. If instead we consider W of Example 7, we have strict dominance of $\lambda(W) = 1$, but with multiplicity 2. Also here no consensus is achieved for the system (3). □

Apart from consensus, when $W \geq 0$, a closely related version of (3) is commonly used to describe transition probabilities in Markov chains. Clearly when W is not nonnegative then any probabilistic interpretation associated to W is lost. However, if $W^t > 0$ for $t \geq t_0$ (i.e., W is EP), any sufficiently long downsampling of the system (3) can still be considered a well-posed transition matrix, provided W is eventually stochastic.

1) Signed undirected graph case: The following necessary and sufficient condition is the analogue of Theorem 3 for discrete-time systems.

Theorem 5 *Consider an undirected signed graph $\mathcal{G}$ of signed adjacency matrix W which is symmetric, and such that $W\mathbf{1} = \mathbf{1}$. W is marginally Schur stable with $\lambda(W) = 1$ simple and strictly dominant iff W is EDS.*

The proof is in Appendix A.

2) Signed digraph case: As for Laplacians, on signed digraphs the sufficient condition for consensus that can be obtained from ES matrices is in general not a necessary condition.

Theorem 6 *Consider a signed digraph $\mathcal{G}$ of adjacency matrix W such that $W\mathbf{1} = \mathbf{1}$. If W is ES, then it is marginally Schur stable with $\lambda(W) = 1$ simple and strictly dominant, and the system* (3) *converges to consensus. Viceversa, if W is marginally Schur stable with $\lambda(W) = 1$ simple and strictly dominant, then $W \in \mathcal{PF}$.*

The proof is in Appendix A.

The condition of eventual stochasticity in Theorem 6 is not necessary for marginal stability with $\lambda(W) = 1$ simple and strictly dominant (see e.g. Example 4 in [13]), while instead the weaker condition $W \in \mathcal{PF}$ does not suffice for marginal stability (see e.g. Example 5 in [13]).

Also for signed stochastic matrices the weight balanced case (here $W\mathbf{1} = W^\top\mathbf{1} = \mathbf{1}$) leads instead to a necessary and sufficient condition.

Corollary 3 *Consider a signed digraph $\mathcal{G}(W)$ such that W is weight balanced and $W\mathbf{1} = \mathbf{1}$. Then the following conditions are equivalent:*

 (i) W is EDS;
 (ii) W is marginally Schur stable with $\lambda(W) = 1$ simple and strictly dominant;

Furthermore, if W is normal then (i) and (ii) are equivalent to

(iii) $I - W^\top W$ is psd of corank 1.

The proof is in Appendix A.

IV. CONSENSUS ON TIME-VARYING SIGNED DIGRAPHS

A. Problem formulation

Given an agent set $\mathcal{V} = [n]$, we consider a group of signed digraphs $\mathbb{G} = \{\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m\}$ over $\mathcal{V}$. We consider two consensus protocols, one in continuous-time and one in discrete-time, in which the time-varying system is approximated as a switching of the underlying graph over $\mathbb{G}$.

1. **Continuous-time protocol**. For each graph $\mathcal{G}_k$, the associated signed Laplacian matrix L_k is defined as in (1). The corresponding set of signed Laplacian matrices is denoted $\mathbb{L} = \{L_1, L_2, \ldots, L_m\}$. The system can be written as

$$\dot{\mathbf{x}} = -L_{\sigma(t)}\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n, \qquad (4)$$

   where $\sigma(\cdot)$ is a switching signal, which is a piecewise constant map, i.e., $\sigma : \mathbb{R}_\geq \mapsto [m]$. Piecewise constant means that any finite interval of $\mathbb{R}_\geq$ can have at most finitely many discontinuities, meaning that Zeno-like phenomena are avoided [19].

2. **Discrete-time protocol**. The corresponding set of signed weighted adjacency matrices is denoted $\mathbb{W} = \{W_1, W_2, \ldots, W_m\}$, where

$$W_k\mathbf{1} = \mathbf{1}, \quad \forall k \in [m].$$

   The discrete-time system is

$$\mathbf{x}(t+1) = W_{\sigma(t)}\mathbf{x}(t), \quad t \in \mathbb{Z}_\geq, \ \mathbf{x}(t) \in \mathbb{R}^n, \qquad (5)$$

   where $\sigma : \mathbb{Z}_\geq \mapsto [m]$ is also a switching signal with the same properties.

For the time-varying systems (4) and (5), in order to achieve consensus in the sense of Definition 7 for an arbitrary switching signal $\sigma(\cdot)$, it is convenient to introduce the following notion of consensus set.

Definition 8 (Consensus set) *The set $\mathbb{L}$ (resp. $\mathbb{W}$) is said to be a consensus set for the system* (4) *(resp.* (5)*) if consensus is achieved for any arbitrary piecewise constant switching signal $\sigma(\cdot)$.*

Problem of interest for this section: find conditions under which the set $\mathbb{L}$ (resp. $\mathbb{W}$) is a consensus set for the system (4) (resp. (5)).

The rest of this section contains our main results for both continuous-time and discrete-time time-varying protocols. Several of these results were first presented in [32]. Here they are rearranged and streamlined in a more efficient way. Only the digraph case is treated in detail, as the undirected graph case follows trivially.

B. Continuous-time protocol

In this subsection we first provide two necessary conditions for consensus sets on digraphs (Lemma 8 and Corollary 4), followed by a sufficient condition (Theorem 7).

For the system (4), in order to achieve consensus under any switching signal, each subsystem $\dot{x} = -L_k x$ must achieve consensus. This observation leads to the following necessary condition, whose proof is a straightforward application of Lemma 4.

Lemma 8 *If $\mathbb{L}$ is a consensus set for the system* (4)*, then $-L_k$ is marginally stable of corank 1 for all $k \in [m]$.*

In the following we will focus on weight balanced digraphs, i.e., $L_k^\top \mathbf{1} = 0, k \in [m]$. As argued above, in order to make $\mathbb{L}$ a consensus set, each L_k must be marginally stable of corank 1. In addition, if $\mathcal{G}_k$ is weight balanced, by Corollary 1 and Lemma 5, the following corollary of Lemma 8 is obtained.

Corollary 4 *Suppose that each $\mathcal{G}_k \in \mathbb{G}$ is weight balanced. If $\mathbb{L}$ is a consensus set, then for all $k \in [m]$, $-L_k$ is EEP, which implies that $\mathcal{G}_k$ is strongly connected.*

However, only EEP and weight balance of each L_k is not enough. This can be seen from the following example.

Example 9 Consider the system (4) on $\mathbb{G} = \{\mathcal{G}_1, \mathcal{G}_2\}$, with the following Laplacians

$$
L_1 = \begin{bmatrix} 0.23 & 0 & -0.28 & 0.05 \\ -0.01 & 0.03 & 0.02 & -0.04 \\ 0.05 & -0.03 & 0.04 & -0.06 \\ -0.27 & 0 & 0.22 & 0.05 \end{bmatrix},
$$

$$
L_2 = \begin{bmatrix} 0.96 & -0.27 & 0 & -0.69 \\ -0.44 & 0.44 & 0 & 0 \\ -0.43 & -0.17 & 0.07 & 0.53 \\ -0.09 & 0 & -0.07 & 0.16 \end{bmatrix}.
$$

L_1, L_2 are both weight balanced. Moreover, 0 is a simple rightmost eigenvalue of $-L_1, -L_2$, which by Corollary 1 means that $-L_1, -L_2$ are EEP. Let the switching signal be

$$
\sigma(t) = \begin{cases} 1, & \text{if } t \in [2d, 2d+1); \\ 2, & \text{if } t \in [2d+1, 2d+2), \end{cases} \quad d = 0, 1, \dots
$$

Let the initial state be $\mathbf{x}(0) = [-1, 2, -4, 7]^\top$. Fig. 1 shows the trajectory of $(x_1(t), x_4(t))$. It is seen that the switching system diverges. □

One condition we can add to EEP is normality of the signed Laplacians.
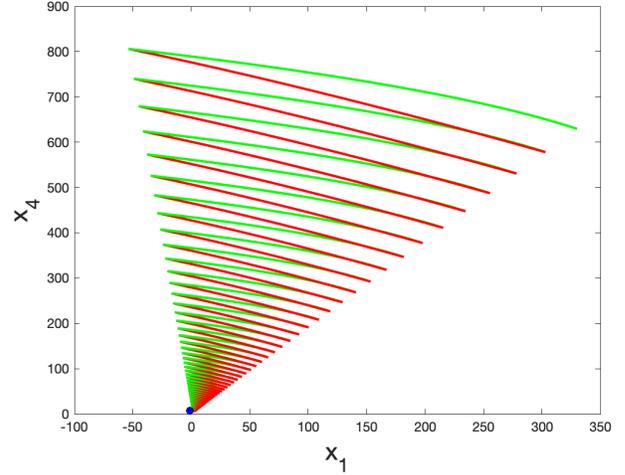
Fig. 1. Two-dimensional slice of a trajectory for Example. 9. The dark dot represents the initial state. Line color changes every time the system switches.

Remark 3 Given a normal matrix, any real right eigenvector is also a left eigenvector corresponding to the same eigenvalue [38]. Thus, any normal signed Laplacian is weight balanced.

The following theorem is the main result of this Section.

Theorem 7 *Consider a set of signed digraphs $\mathbb{G} = \{\mathcal{G}_1, \dots, \mathcal{G}_m\}$ with the corresponding Laplacians $\mathbb{L} = \{L_1, \dots, L_m\}$. If $-L_k$ is a EEP, normal matrix for each $k \in [m]$, then $\mathbb{L}$ is a consensus set for the system* (4)*.*

The proof is in Appendix B.

Remark 4 Any symmetric matrix is normal. Therefore, if for all $k \in [m]$, $-L_k$ is symmetric and EEP, $\mathbb{L}$ is then a consensus set.

Note that the proof of Theorem 7 is similar to that in [39]. The requirement that $L_k + L_k^\top, k \in [m]$, is psd of corank 1 is key to the proof. This condition is equivalent to the existence of a common quadratic Lyapunov function with matrix which is diagonal and equal to I. To satisfy this condition, the normality of each Laplacian is sufficient but not necessary, see Sect. V for a counterexample.

One may also ask if the condition $L_k + L_k^\top$ is psd of corank 1 for all $k \in [m]$ is a necessary condition. The answer is no, see the following example.

Example 10 Consider the system (4) on $\mathbb{G} = \{\mathcal{G}_1, \mathcal{G}_2\}$ with the corresponding Laplacians L_1 given in Example 9 and $L_2 = I - \frac{1}{4}\mathbf{1}\mathbf{1}^\top$. It can be checked that $L_1 + L_1^\top$ is not psd. For any given switching signal $\sigma(t)$, without loss of generality, suppose the signals switch in the sequence $1, 2, 1, 2, \dots$ with switching time $0 < t_1 < t_2 < t_3 < \dots$. Let

$$
t_1 = s_1, t_2 - t_1 = \tau_1, t_3 - t_2 = s_2, t_4 - t_3 = \tau_2, \dots
$$

The solution can be explicitly written as

$\mathbf{x}(t) =$

$$
\begin{cases}
[(1 - e^{-\sum\limits_{d=1}^{\kappa}\tau_d})\frac{1}{n}\mathbf{1}\mathbf{1}^\top + e^{-\sum\limits_{d=1}^{\kappa}\tau_d}e^{-(s'+\sum\limits_{d=1}^{\kappa}s_d)L_1}]\mathbf{x}(0), \\
\quad \text{if } t = \sum\limits_{d=1}^{\kappa}(s_d+\tau_d) + s', \quad 0 \le s' < s_{\kappa+1}; \\
[(1 - e^{-\tau'-\sum\limits_{d=1}^{\kappa}\tau_d})\frac{1}{n}\mathbf{1}\mathbf{1}^\top + e^{-\tau'-\sum\limits_{d=1}^{\kappa}\tau_d}e^{(-\sum\limits_{d=1}^{\kappa+1}s_d)L_1}]\mathbf{x}(0), \\
\quad \text{if } t = \sum\limits_{d=1}^{\kappa}(s_d+\tau_d) + s_{\kappa+1} + \tau', \quad 0 \le \tau' < \tau_{\kappa+1}.
\end{cases}
$$

Then

$$
\lim_{t\to\infty}\mathbf{x}(t) = [(1 - e^{-\sum\limits_{d=1}^{\infty}\tau_d})\frac{1}{n}\mathbf{1}\mathbf{1}^\top + e^{-\sum\limits_{d=1}^{\infty}\tau_d}e^{-(\sum\limits_{d=1}^{\infty}s_d)L_1}]\mathbf{x}(0). \tag{6}
$$

It is easy to see that:

- if $\sum_{d=1}^{\infty}\tau_d = \infty$, the second part of (6) is 0, which yields $\lim_{t\to\infty}\mathbf{x}(t) = \frac{1}{n}\mathbf{1}\mathbf{1}^\top\mathbf{x}(0)$;
- if $\sum_{d=1}^{\infty}\tau_d \neq \infty$, it must be $\sum_{d=1}^{\infty}s_d = \infty$. We then have

$$
e^{-(\sum\limits_{d=1}^{\infty}s_d)L_1} = \frac{1}{n}\mathbf{1}\mathbf{1}^\top,
$$

since $-L_1$ is EEP. According to (6), it holds $\lim_{t\to\infty}\mathbf{x}(t) = \frac{1}{n}\mathbf{1}\mathbf{1}^\top\mathbf{x}(0)$.

No matter in which case, it always holds that $\lim_{t\to\infty}\mathbf{x}(t) = \frac{1}{n}\mathbf{1}\mathbf{1}^\top\mathbf{x}(0)$. As $\sigma(t)$ is arbitrarily chosen, $\mathbb{L} = \{L_1, L_2\}$ is a consensus set. $\square$

C. Discrete-time protocol

The discrete-time case is investigated analogously to the continuous-time one: two necessary conditions for consensus sets are presented first (Lemma 9 and Corollary 5), followed by a sufficient condition (Theorem 8).

For the discrete-time system (5), in order to achieve consensus under any switching signal, each subsystem $\mathbf{x}(t+1) = W_k\mathbf{x}(t)$ must achieve consensus, which leads to the following necessary condition.

Lemma 9 Suppose $W_k\mathbf{1} = \mathbf{1}$ for all $k \in [m]$. If $\mathbb{W}$ is a consensus set for the system (5), then for all $k \in [m]$, W_k is marginally Schur stable with $\lambda(W_k) = 1$ as a simple and strictly dominant eigenvalue.

The lemma is proven straightforwardly from Lemma 7. From Theorem 6, it implies that it must be $W_k \in \mathcal{PF}\ \forall k \in [m]$. In the weight balanced case, combining Lemma 9 with Lemma 6 a corollary follows.

Corollary 5 Suppose that each $\mathcal{G}_k \in \mathbb{G}$ is weight balanced. If $\mathbb{W}$ is a consensus set, then for all $k \in [m]$, W_k is EDS and hence EP, which implies that each $\mathcal{G}_k$ is strongly connected.

However, just like in the continuous-time case, only EP/EDS is still not enough, as the combined switching system may still diverge, see e.g. Example 3 of [32].

Theorem 8 Consider a signed digraph set $\mathbb{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_m\}$ with the corresponding weighted adjacency matrices $\mathbb{W} =$ $\{W_1, \ldots, W_m\}$. If, for all $k \in [m]$, W_k is a ES and normal matrix, then $\mathbb{W}$ is a consensus set for the system (5).

The proof is in Appendix B.

Remark 5 As in Remark 4, if W_k is symmetric and ES for all $k \in [m]$, $\mathbb{W}$ is a consensus set.

According to the proof of Theorem 8, we can see that it is enough to have $I - W_k^\top W_k$ psd and of corank 1 for all $k \in [m]$. Therefore, the condition that each W_k is normal is sufficient but not necessary, which is also shown in the following example.

Example 11 Let $\mathbb{G} = \{\mathcal{G}_1, \mathcal{G}_2\}$ corresponding to $\mathbb{W} = \{W_1, W_2\}$, with

$$
W_1 = \begin{bmatrix} 0.1 & 0.2 & 0.7 \\ 0.3 & 0.5 & 0.2 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}, \quad W_2 = W_1^\top.
$$

W_1 and W_2 are positive and doubly stochastic. Obviously $\mathbb{W}$ is a consensus set for the system (5) (since $\mathbb{W}$ is a finite set of stochastic matrices with positives columns [40]), but W_1 and W_2 are not normal. $\square$

Again, also the condition that $I - W_k^\top W_k$ is psd and of corank 1 for all $k \in [m]$ is not necessary, see e.g. Example 4 of [32].

V. MORE GENERAL SUFFICIENT CONDITIONS FOR THE TIME-VARYING CASE

It is well-known [18], [19] that a sufficient condition for a collection of Hurwitz (or Schur) stable linear systems to be stable as a switching system is to have a Common Lyapunov Function (CLF). As the proofs show, also the conditions of Theorems 7 and 8 are based on the existence of a CFL, which is diagonal and with diagonal weights all equal to 1. While such sufficient condition is of interest per se, as it has an equivalent algebraic formulation in terms of EEP/ES normal matrices, it can be relaxed to any (non-diagonal) CLF, allowing us to relax the assumption of normality of the L_k and W_k. In this way, the class of switching signed graphs for which time-varying consensus is achieved is enlarged significantly. Existence of a CLF can be expressed as feasibility of a system of Linear Matrix Inequalities (LMIs).

In what follows we present a continuous-time LMI condition for consensus sets based on quadratic CLFs (Theorem 9) and one based on CLFs which are homogeneous polynomials of order higher than 2 (Corollary 6), followed by the same conditions for discrete-time systems (Theorem 10 and Corollary 7).

A. Continuous-time case

Denote $Q \in \mathbb{R}^{n-1 \times n}$ a matrix whose rows form an orthonormal basis for $\text{span}(\mathbf{1})^\perp$, i.e.,

$$
QQ^\top = I_{n-1}, \quad Q\mathbf{1} = \mathbf{0}.
$$

Theorem 9 *Consider a set of signed digraphs* $\mathbb{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_m\}$ *with the corresponding Laplacians* $\mathbb{L} = \{L_1, \ldots, L_m\}$. *Assume* $-L_k$ *is a weight balanced EEP matrix for each* $k \in [m]$. *If* $\exists\, P = P^\top \succ 0$ *such that*

$$-QL_k PQ^\top - QPL_k^\top Q^\top \prec 0, \quad k = 1, \ldots, m \quad (7)$$

then $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$ *is a CLF and hence* $\mathbb{L}$ *is a consensus set for the system* (4).

The proof is in Appendix B.

Example 12 None of the following two weight-balanced Laplacian matrices whose negation is EEP

$$L_1 = \begin{bmatrix} 0.3388 & -0.5673 & 0.2285 \\ 0 & 0.5673 & -0.5673 \\ -0.3388 & 0 & 0.3388 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 0.3937 & -0.6570 & 0.2633 \\ -0.1862 & 0.6570 & -0.4708 \\ -0.2075 & 0 & 0.2075 \end{bmatrix}$$

is normal. For these Laplacians, taking

$$P = \begin{bmatrix} 72.7760 & 18.9053 & -6.5237 \\ 18.9053 & 68.4564 & -2.2041 \\ -6.5237 & -2.2041 & 93.8854 \end{bmatrix},$$

$V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$ is a CLF. $\square$

Similarly to what happens to switching systems of Hurwitz matrices, existence of a quadratic CLF is sufficient but not necessary, and higher order homogeneous polynomial CLF can be used to relax the conservatism of quadratic design [26]. Following [41], to build a $2r$-th order homogeneous CLF we can make use of (r-times) Kronecker products and Kroneker sums, defining

$$\hat{\mathbf{x}} = \mathbf{x} \otimes \ldots \otimes \mathbf{x}$$
$$\hat{L}_k = L_k \oplus \ldots \oplus L_k$$
$$\hat{Q} = Q \otimes \ldots \otimes Q.$$

Easy calculations then give the following extension of Theorem 9.

Corollary 6 *Consider a set of signed digraphs* $\mathbb{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_m\}$ *with the corresponding Laplacians* $\mathbb{L} = \{L_1, \ldots, L_m\}$. *Assume* $-L_k$ *is a weight balanced EEP matrix for each* $k \in [m]$. *If* $\exists\, \hat{P} \in \mathbb{R}^{n^r \times n^r}$, $\hat{P} = \hat{P}^\top \succ 0$ *such that*

$$-\hat{Q}\hat{L}_k\hat{P}\hat{Q}^\top - \hat{Q}\hat{P}\hat{L}_k^\top\hat{Q}^\top \prec 0, \quad k = 1, \ldots, m$$

then $V(\mathbf{x}) = \hat{\mathbf{x}}^\top \hat{P} \hat{\mathbf{x}}$ *is a CLF and hence* $\mathbb{L}$ *is a consensus set for the system* (4).

Example 13 The following two signed Laplacian matrices

$$L_1 = \begin{bmatrix} 0.6667 & 0.2440 & -0.9107 \\ -0.9107 & 0.6667 & 0.2440 \\ 0.2440 & -0.9107 & 0.6667 \end{bmatrix}$$

$$L_2 = \begin{bmatrix} 2.6111 & 0.4746 & -3.0857 \\ -3.0857 & -0.3056 & 3.3913 \\ 0.4746 & -0.1691 & -0.3056 \end{bmatrix}$$

do not admit a quadratic CLF, but they admit a CLF which is a homogeneous polynomial of order 4. In fact their projections $\bar{L}_k = QL_kQ^\top$ correspond to a famous example of 2D switching system not admitting a quadratic CLF [42]:

$$\bar{L}_1 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad \bar{L}_2 = \begin{bmatrix} 1 & 6 \\ -1/6 & 1 \end{bmatrix}.$$

$\square$

B. Discrete-time case

Also for the discrete-time case the consensus set stability can be extended beyond normality via CLF.

Theorem 10 *Consider a set of signed digraphs* $\mathbb{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_m\}$ *with the corresponding EDS matrices* $\mathbb{W} = \{W_1, \ldots, W_m\}$. *If* $\exists\, P = P^\top \succ 0$ *such that*

$$QW_k^\top PW_kQ^\top - QPQ^\top \prec 0, \quad k = 1, \ldots, m \quad (8)$$

then $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$ *is a CLF and hence* $\mathbb{W}$ *is a consensus set for the system* (5).

The proof is in Appendix B.

Denoting

$$\hat{W}_k = W_k \otimes \ldots \otimes W_k,$$

we obtain also the discrete-time equivalent of Corollary 6.

Corollary 7 *Consider a set of signed digraphs* $\mathbb{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_m\}$ *with the corresponding EDS matrices* $\mathbb{W} = \{W_1, \ldots, W_m\}$. *If* $\exists\, \hat{P} \in \mathbb{R}^{n^r \times n^r}$, $\hat{P} = \hat{P}^\top \succ 0$ *such that*

$$\hat{Q}\hat{W}_k^\top\hat{P}\hat{W}_k\hat{Q}^\top - \hat{Q}\hat{P}\hat{Q}^\top \prec 0, \quad k = 1, \ldots, m$$

then $V(\mathbf{x}) = \hat{\mathbf{x}}^\top \hat{P} \hat{\mathbf{x}}$ *is a CLF and hence* $\mathbb{W}$ *is a consensus set for the system* (5).

Example 14 The following two signed EDS matrices

$$W_1 = \begin{bmatrix} 0.2000 & 0.1691 & 0.6309 \\ 0.6309 & 0.2000 & 0.1691 \\ 0.1691 & 0.6309 & 0.2000 \end{bmatrix}$$

$$W_2 = \begin{bmatrix} -0.5778 & 0.0768 & 1.5010 \\ 1.5010 & 0.5889 & -1.0898 \\ 0.0768 & 0.3343 & 0.5889 \end{bmatrix}$$

do not admit a quadratic CLF, but they admit a CLF which is a homogeneous polynomial of order 4. In this case their projections $\bar{W}_k = QW_kQ^\top$ correspond to another known example of uniformly stable switching system not admitting a quadratic CLF [43]:

$$\bar{W}_1 = \begin{bmatrix} -0.2 & -0.4 \\ 0.4 & -0.2 \end{bmatrix}, \qquad \bar{W}_2 = \begin{bmatrix} -0.2 & -2.4 \\ 1/15 & -0.2 \end{bmatrix}.$$

$\square$

VI. EXTENSION TO BIPARTITE CONSENSUS

Consider a signed digraph $\mathcal{G}$. Given a diagonal signature matrix $S = \mathrm{diag}(\mathbf{s})$, consider the following generalizations of L and W:

$$L_b = SLS \quad \text{and} \quad W_b = SWS, \tag{9}$$

and the corresponding generalizations of (2) or (3):

$$\dot{\mathbf{x}} = -L_b \mathbf{x} \tag{10}$$

and

$$\mathbf{x}(t+1) = W_b \mathbf{x}(t). \tag{11}$$

Notice that L_b does not obey to the rule (1). In fact, $L_b = SLS = \Sigma - SAS$, and in general $\sigma_i^{\mathrm{in}} = \sum_{j=1}^n A_{ij} \neq \sum_{j=1}^n [\mathbf{s}]_i [\mathbf{s}]_j A_{ij}$. In matrix form, this reads $L_b \mathbf{s} = 0$. Similarly, in place of $W\mathbf{1} = \mathbf{1}$, we now have $W_b \mathbf{s} = \mathbf{s}$.

The systems (10) or (11) are associated to a more general form of consensus, called bipartite consensus [1].

Definition 9 *We say that the system* (10) *or* (11) *achieves bipartite consensus if for all* $\mathbf{x}(0) \in \mathbb{R}^n$, *there exist* $\alpha \in \mathbb{R}$ *and a diagonal signature matrix* $S = \mathrm{diag}(\mathbf{s})$ *such that* $\mathbf{x}^* = \lim_{t \to \infty} \mathbf{x}(t) = \alpha \mathbf{s}$.

Bipartite consensus corresponds to all agents achieving the same value in absolute value but possibly with different sign: $[\mathbf{x}^*]_i = \pm\alpha$, see [1]. In particular, each signature vector $\mathbf{s}$ determines a "bipartition class", i.e., a splitting of the n agents into two disjoint subgroups (up to a global symmetry $\mathbf{s} \to -\mathbf{s}$).

Several properties of L_b and W_b follow straightforwardly from (9) and Proposition 1. For instance, for L_b we have:

Proposition 2 *Given a signed digraph* $\mathcal{G}$, *of Laplacian* L, *consider the associated matrix* L_b *in* (9) *corresponding to* $S = \mathrm{diag}(\mathbf{s})$ *and the system* (10). *Then we have:*

1) *0 is always an eigenvalue of* L_b *of right eigenvector* $\mathbf{s}$;
2) $\mathbf{x}(t)$ *converges to* $\alpha\mathbf{s}$ *iff* $\mathbf{z}(t) = S\mathbf{x}(t)$ *converges to* $\alpha\mathbf{1}$, *i.e., bipartite consensus is achieved by* $\mathbf{x}(t)$ *iff consensus is achieved by* $\mathbf{z}(t)$.
3) *bipartite consensus is achieved iff* $-L_b$ *is marginally stable of corank 1.*

The proof is in Appendix B.

All sufficient conditions for consensus given in the previous sections can be extended to bipartite consensus, provided we replace the notions of EP, EEP and ES with SEP, SEEP and SES. For instance the equivalent of Theorem 4 and Corollary 1 is the following.

Theorem 11 *Consider a signed digraph* $\mathcal{G}$ *and a diagonal signature matrix* $S = \mathrm{diag}(\mathbf{s})$.

- *If* $-L_b$ *is SEEP (w.r.t. S) then* $-L_b$ *is marginally stable of corank 1; Viceversa, if* $-L_b$ *is marginally stable of corank 1 then* $\exists d \geq 0$ *such that* $dI - L_b \in \mathcal{SPF}_n$.
- *If, in addition,* $L_b \mathbf{s} = L_b^\top \mathbf{s} = 0$, *then the following two conditions are equivalent:*
  (i) $-L_b$ *is SEEP (w.r.t. S);*
  (ii) $-L_b$ *is marginally stable of corank 1;*

- *If, in addition,* L_b *is normal, then* (i) *and* (ii) *above are equivalent to*
  (iii) $(L_b + L_b^\top)/2$ *is psd of corank 1.*

The proof follows straightforwardly once the change of basis $\mathbf{z}(t) = S\mathbf{x}(t)$ is performed.

Remark 6 Recall that a signed digraph is said structurally balanced if all its directed cycles are positive (i.e., have an even number of negative edges). As shown in [1], for the "opposing" signed Laplacian structural balance of the digraph is a necessary and sufficient condition for bipartite consensus. The situation is very different for the "repelling" signed Laplacian L and its generalization L_b we are considering in this paper. L can attain consensus even in presence of negative cycles, and it never achieves bipartite consensus. L_b can attain bipartite consensus even in presence of negative cycles, but it can also do so when all cycles are positive (in that case the corresponding L is nonnegative). Cycle sign is irrelevant for the "repelling" Laplacian L and its generalization L_b; the only thing that matters is the signature of the right PF eigenvector $\mathbf{s}$.

Also the time-varying case can be treated analogously, provided that all switching systems correspond to the same signature matrix $S = \mathrm{diag}(\mathbf{s})$. If we have the collection of signed digraphs $\mathbb{G} = \{\mathcal{G}_1, \ldots, \mathcal{G}_m\}$ of Laplacians $\mathbb{L} = \{L_1, \ldots, L_m\}$, then consider the generalizations $\mathbb{L}_b = \{L_{b,1}, \ldots, L_{b,m}\}$, $L_{b,k} = SL_k S$, $k \in [m]$, and the system

$$\dot{\mathbf{x}} = -L_{b,\sigma(t)} \mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^n, \tag{12}$$

where $\sigma : \mathbb{R}_{\geq} \mapsto [m]$ is a piecewise constant switching signal.

Definition 10 (Bipartite consensus set) *The set* $\mathbb{L}_b$ *is said to be a bipartite consensus set for the system* (12) *w.r.t. the signature* $\mathbf{s}$ *if bipartite consensus with node partition given by* $\mathbf{s}$ *is achieved for any arbitrary piecewise constant switching signal* $\sigma(\cdot)$.

Proposition 3 *The set* $\mathbb{L}_b$ *is a bipartite consensus set for the system* (10) *w.r.t. the signature* $\mathbf{s}$ *iff the set* $\mathbb{L}$ *is a consensus set for the system* (2).

Again the proof is immediate with the change of basis $\mathbf{z}(t) = S\mathbf{x}(t)$. All conditions of Sections V and VI can then be translated into equivalent conditions for bipartite consensus for $\mathbb{L}_b$.

A similar reasoning applies also to the discrete-time case.

VII. DISCUSSION AND OUTLOOK

In this section we comment on a couple of possible developments of our work.

a) Algebraic vs graphical conditions: A remarkable fact about the "opposing" signed Laplacian studied in [1] is that its main properties are both *algebraic and graphical*. Following [44], we can define qualitative classes of signed adjacency matrices only based on their sign patterns: the matrices A_1 and A_2 are said to belong to the same qualitative class $\mathcal{Q}[A_1]$ (or $\mathcal{Q}[A_2]$) if $\operatorname{sgn}([A_1]_{ij}) = \operatorname{sgn}([A_2]_{ij}) \; \forall i,j = 1,\ldots,n$. In particular, all matrices of a qualitative class share the same property of being – or not being – structurally balanced. Consequently, all associated "opposing" signed Laplacians in the qualitative class either converge to bipartite consensus (when the class is structurally balanced; all L are marginally stable) or converge to the origin (when the class is unbalanced; all L are Hurwitz). The consensus and bipartite consensus conditions considered in this paper for "repelling" signed Laplacians are instead *algebraic but not graphical*, as they are decided by the dominant eigenvalue and by the signature of the dominant eigenvector. It is rather easy to find counterexamples of matrices in the same qualitative class (i.e., with identical sign pattern) but with different asymptotic behaviors. For instance, if in the unstable signed Laplacian of Example 3 the $(1,3)$ entry 0.5 is replaced with 0.1, then the corresponding signed Laplacian becomes marginally stable of corank 1, meaning that consensus is achieved. Such counterexamples are found also for undirected signed graphs. Similarly, showing that a signed Laplacian converges to bipartite consensus instead of consensus is at the moment impossible in graphical terms, without computing the signature of the dominant eigenvector. The conclusion is that sharp equivalent graphical characterizations of all major properties of "repelling" signed Laplacians do not appear to exist. Except for special graphs (e.g. trees), what one can find instead are bounds on the number of positive/negative eigenvalues, based on the number of negative edges, see [9], [3], [45]. How effective those bounds are is not clear at the moment.

b) Time-varying consensus beyond consensus set: In Sections IV and V of the paper, time-varying consensus is always formulated in terms of consensus sets, i.e., sets $\mathbb{L}$ such that for *any* switching signal $\sigma(\cdot)$, $L_{\sigma(t)} \in \mathbb{L}$ is an acceptable time-varying Laplacian. A weaker condition could be instead to try to assess consensus for a *specific* switching pattern $\sigma(\cdot)$. The sufficient conditions of Theorem 7 may be potentially weakened in this case, in a way similar to what happens for "nonnegative" consensus problems. However, differently from the "nonnegative" consensus case, switching signed Laplacians may lead to instability and not just to lack of convergence, meaning that treating the specific switching pattern case under weaker conditions is a nontrivial problem. It might require to define a notion of "uniform marginal stability of corank 1" and to find conditions implementing it in practice.

VIII. Conclusions

The conditions provided in this paper for checking the stability of signed ("repelling") Laplacians are more general than those available in the literature, as they hold also for digraphs, and for time-varying signed Laplacians. They are also insightful into the structure of these Laplacians, as they

highlight the role of the dominant eigenpair of L. As computing dominant eigenpairs (through e.g. a power algorithm) is computationally cheaper than computing the entire spectrum of L, the tests provide also a computational advantage for large scale systems.

Also in the time-varying case the behavior of signed Laplacians is qualitatively different from that of its standard "nonnegative" counterpart, in the sense that instabilities may arise due to the time dependence, making it even more relevant to seek some form of uniform convergence when dealing with this class of time-varying consensus problems.

Appendix A
Time-invariant digraphs

Proof of Proposition 1. Property 1 is true by construction. Properties 2-5 can be shown through counterexamples (Examples 1-3). $\qquad\square$

Proof of Lemma 4. We first prove the necessity. Notice that 0 is an eigenvalue of L, with $\mathbf{1}$ included in its corresponding eigenvector space. The system $\dot{\mathbf{x}} = -L\mathbf{x}$ has the explicit solution $\mathbf{x}(t) = e^{-Lt}\mathbf{x}(0)$. If consensus is achieved, it must be $\lim_{t\to\infty} e^{-Lt} = \mathbf{1}\mathbf{c}^\top$ for some $\mathbf{c} \in \mathbb{R}^n$. Obviously $-L$ can not have eigenvalues with positive real parts, otherwise e^{-Lt} will diverge. To prove marginal stability of $-L$, we use a contradiction argument. First, assume that 0 is not a simple root of the minimal polynomial of $-L$. Without loss of generalization, assume that L can be decomposed to $L = CJC^{-1}$, where $J = \operatorname{diag}(\begin{bmatrix} J_1 & J_2 & \ldots & J_s \end{bmatrix})$ is the Jordan canonical form of L, with blocks $J_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $J_2,\ldots,J_s$ corresponding to eigenvalues of L with positive real part. It then holds $e^{-Lt} = Ce^{-Jt}C^{-1}$, with $e^{-Jt} = \operatorname{diag}(\begin{bmatrix} e^{-J_1 t} & e^{-J_2 t} & \ldots & e^{-J_s t} \end{bmatrix})$. By straightforward calculations, we obtain

$$e^{-J_1 t} = \begin{bmatrix} 1 & -t \\ 0 & 1 \end{bmatrix}$$

and

$$\lim_{t\to\infty} \operatorname{diag}(\begin{bmatrix} (e^{-J_2 t} & \ldots & e^{-J_s t} \end{bmatrix}) = \mathbf{0}.$$

Therefore,

$$\lim_{t\to\infty} e^{-Lt} = \lim_{t\to\infty} C[e^{-Jt}]C^{-1} = \mathbf{1}\mathbf{c}^\top$$

is impossible in presence of the term $e^{-J_1 t}$. We then get the contradiction. Second, suppose that $di \in \Lambda(-L)$ for some $d \in \mathbb{R}$, where i is the imaginary unit. Without loss of generality, the Jordan canonical form of L then has a block J_1 such that $J_1 = \begin{bmatrix} di & 1 \\ 0 & di \end{bmatrix}$ or $J_1 = \begin{bmatrix} di & 0 \\ 0 & -di \end{bmatrix}$. Therefore,

$$\lim_{t\to\infty} e^{-J_1 t} = \begin{bmatrix} e^{-dti} & * \\ 0 & e^{-dti} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} e^{-dti} & 0 \\ 0 & e^{dti} \end{bmatrix},$$

where $*$ can be infinity. Again, no matter what case, it can not be $\lim_{t\to\infty} e^{-Lt} = \mathbf{1}\mathbf{c}^\top$, leading again to a contradiction. Therefore, $-L$ is marginally stable. To prove in

addition that L is of corank 1, we use another contradiction argument. Suppose that $-L$ is marginally stable of corank greater than or equal to 2. Let $J = CJC^{-1}$, where $J = \text{diag}([(J_1 \quad \dots \quad J_{\text{corank}(L)} \quad J_{\text{corank}(L)+1} \quad \dots \quad J_s])$ is the Jordan canonical form of L, with $J_\ell = [0]$ for all $\ell = 1, \dots, \text{corank}(L)$, and J_ℓ, $\ell > \text{corank}(L)$, corresponding to eigenvalues of L with positive real parts. Then,

$$\lim_{t \to \infty} e^{-Lt} = \sum_{\ell=1}^{\text{corank}(L)} \phi_\ell \boldsymbol{\xi}_\ell^\top \neq \mathbf{1}\mathbf{c}^\top, \tag{13}$$

where $\phi_\ell, \boldsymbol{\xi}_\ell^\top$ are respectively the columns and rows of C and C^{-1}. The contradiction is then obtained and the necessity is proved. The sufficiency obviously follows from (13). $\square$

Proof of Theorem 4. From Lemma 2, $-L$ EEP means that $\exists$ a scalar $d \geq 0$ such that $B = dI - L \overset{\vee}{>} 0$. From $L\mathbf{1} = 0$, it is $B\mathbf{1} = d\mathbf{1} - L\mathbf{1} = d\mathbf{1}$, i.e., $\mathbf{1}$ is a right eigenvector of B of eigenvalue d. If $B \overset{\vee}{>} 0$ it means that $\rho(B)$ is a simple eigenvalue of strictly positive left and right eigenvectors $\mathbf{v}_r$ and $\mathbf{v}_\ell$. From Lemma 1, it must necessarily be $\mathbf{v}_r = \alpha\mathbf{1}$ for some scalar α, and therefore $\rho(B) = d$. Since $\rho(B)$ is simple, all other eigenvalues of L must have strictly positive real parts. The value of $\mathbf{x}^*$ follows then straightforwardly. As for the second implication, if $-L$ is marginally stable and of corank 1, then $0 = \lambda_1(L) < \text{Re}[\lambda_i(L)]$, $i = 2, \dots, n$, and $\mathbf{1}$ is the eigenvector relative to 0. But then, provided $d > \max_{i=2,\dots,n} \frac{|\lambda_i(L)|^2}{2\text{Re}[\lambda_i(L)]}$, $B = dI - L$ has $\rho(B) = d$ of eigenvector $\mathbf{1}$, hence $B \in \mathcal{PF}$. $\square$

Proof of Corollary 1. (i) $\implies$ (ii): This direction is implied by Theorem 4.
(ii) $\implies$ (i): If $-L$ marginally stable of corank 1, it follows from the proof of Theorem 4 that for $d > \max_{i=2,\dots,n} \frac{|\lambda_i(L)|^2}{2\text{Re}[\lambda_i(L)]} > 0$, $B = dI - L$ has $\rho(B) = d$ which is a simple eigenvalue of right eigenvector $\mathbf{1}$. L weight balanced implies $L\mathbf{1} = L^\top\mathbf{1} = 0$, which means that the argument can be repeated also for $L^\top$ leading to B, $B^\top \in \mathcal{PF}$, i.e., from Theorem 1, $B \overset{\vee}{>} 0$ or, from Lemma 2, $-L$ is EEP.
(ii) $\iff$ (iii): If L is normal, then there exists an orthonormal matrix U such that $L = UDU^\top$, where, if $\mu_1, \dots, \mu_\ell$ are the real eigenvalues of L and $\nu_1 \pm i\omega_1, \dots, \nu_{\frac{n-\ell}{2}} \pm i\omega_{\frac{n-\ell}{2}}$ are its complex conjugate eigenvalues:

$$D = \begin{bmatrix} \mu_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \mu_\ell & & & & & \\ & & & \nu_1 & \omega_1 & & & \\ & & & -\omega_1 & \nu_1 & & & \\ & & & & & \ddots & & \\ & & & & & & \nu_{\frac{n-\ell}{2}} & \omega_{\frac{n-\ell}{2}} \\ & & & & & & -\omega_{\frac{n-\ell}{2}} & \nu_{\frac{n-\ell}{2}} \end{bmatrix} \tag{14}$$

If follows that $L_s = \frac{1}{2}(L + L^\top) = \frac{1}{2}U(D + D^\top)U^\top$ and therefore that $\text{Re}[\lambda_i(L)] = \lambda_i(L_s)$. $\square$

Proof of Corollary 2. First, observe that, since L_s is psd, then $\mathbf{x}^\top L_s\mathbf{x} = 0$ iff $L_s\mathbf{x} = 0$ [46, Observation 7.1.6.]. Moreover,

since L_s is of corank 1 and $L\mathbf{1} = 0$ (which implies $\mathbf{1}^\top L_s\mathbf{1} = 0$), then $\ker(L_s) = \text{span}(\mathbf{1})$. Assume by contradiction that L is not weight balanced, i.e., $L^\top\mathbf{y} = 0$ with $\mathbf{y} \neq \mathbf{1}$. Then:

$$0 = \mathbf{y}^\top L^\top\mathbf{y} = \mathbf{y}^\top L_s\mathbf{y} \implies L_s\mathbf{y} = 0 \implies \mathbf{y} = \mathbf{1}$$

since $\ker(L_s) = \text{span}(\mathbf{1})$. Hence, L is weight balanced. Assume by contradiction that $\text{corank}(L) \neq 1$, i.e., $L\mathbf{x} = 0$ with $\mathbf{x} \neq \mathbf{1}$. Similarly to the previous case, it is possible to show that L_s being psd of corank 1 implies $\mathbf{x} = \mathbf{1}$. Since L_s psd implies L has eigenvalues with nonnegative real part, marginal stability of $-L$ follows straightforwardly. $\blacksquare$

Proof of Lemma 5.
1) Let $\mathcal{G}$ be a signed digraph with signed Laplacian L (with at least 2 nodes). First, we establish an equivalent condition for $\mathcal{G}$ to have a rooted spanning tree. In the unsigned graph case it has been proven, see [47, Theorem 5], that $\mathcal{G}$ (unsigned) has a rooted spanning tree if and only if for every pair of nonempty, disjoint subsets $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{V}$ there exists a node that is an in-neighbor of $\mathcal{V}_1$ or $\mathcal{V}_2$ (see also [48]). Since the proof does not depend on the specific weights on the edges of the graph, it holds also in the case of a signed graph $\mathcal{G}$. This means that, for every pair of nonempty, disjoint subsets $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{V}$, after an adequate permutation, L can be rewritten as

$$L = \begin{bmatrix} L_{11} & L_{12} \\ 0 & L_{22} \end{bmatrix}, L_{12} \neq 0, \text{ or } L = \begin{bmatrix} L_{11} & 0 \\ L_{21} & L_{22} \end{bmatrix}, L_{21} \neq 0,$$

assuming an edge from $\mathcal{V}_2$ to $\mathcal{V}_1$, or from $\mathcal{V}_1$ to $\mathcal{V}_2$, respectively.
Now, assume by contradiction that L is of corank 1, but $\mathcal{G}$ does not have a rooted spanning tree. Then, there exists a pair of nonempty, disjoint subsets $\mathcal{V}_1, \mathcal{V}_2 \subset \mathcal{V}$, such that, after an adequate permutation, L can be rewritten as

$$L = \begin{bmatrix} L_{11} & 0 \\ 0 & L_{22} \end{bmatrix}.$$

Then, $L\mathbf{1} = 0$ implies that $0 \in \Lambda(L_{11})$, $0 \in \Lambda(L_{22})$, and $\ker(L) = \text{span}\left(\begin{bmatrix} \mathbf{1}_{\text{card}(\mathcal{V}_1)} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathbf{1}_{\text{card}(\mathcal{V}_2)} \end{bmatrix} \right)$. Consequently, L is not of corank 1.
2) In both statements assume, by contradiction, that L is reducible, i.e., there exists a permutation matrix P s.t. $P^\top LP = \begin{bmatrix} L_{11} & L_{12} \\ 0 & L_{22} \end{bmatrix}$.
Assume that $-L$ is EEP, i.e., $\exists d \geq 0$ s.t. $B = dI - L \overset{\vee}{>} 0$ (see Lemma 2). Then B is also reducible, since $P^\top BP = \begin{bmatrix} dI - L_{11} & -L_{12} \\ 0 & dI - L_{22} \end{bmatrix}$. It follows that $(P^\top BP)^t = \begin{bmatrix} (dI - L_{11})^t & * \\ 0 & (dI - L_{22})^t \end{bmatrix}$ for all $t \geq 1$, i.e., $P^\top BP$ is not EP and, consequently, B is not EP.
Assume that L is weight balanced of corank 1. Then $L\mathbf{1} = L^\top\mathbf{1} = 0$ implies that $0 \in \Lambda(L_{11}^\top) = \Lambda(L_{11})$ and that $0 \in \Lambda(L_{22})$. Consequently, L is not of corank 1. $\square$

Proof of Lemma 7. To prove the necessity, we need to prove that $\rho(W) = 1$ and 1 is a simple and strictly dominating

eigenvalue. As $\mathbf{x}(t) = W^t\mathbf{x}(0)$, we need to analyze W^t as t grows to infinity. From $W\mathbf{1} = \mathbf{1}$, it is $\lambda(W) = 1$, hence $\rho(W) \geq 1$. If $\rho(W) > 1$, it holds $\rho(W^t) = (\rho(W))^t \to \infty$, i.e., W^t diverges as t grows. Therefore, $\rho(W) = 1$. The rest of the proof can be obtained by an argument in the same spirit as that of Lemma 4 for what concerns both the multiplicity of $\lambda(W) = 1$ and its strict dominance. The sufficiency is then obtained repeating the reasoning which led to (13). $\square$

Proof of Theorem 5. W symmetric and marginally Schur stable with $\lambda_1(W) = 1$ simple and strictly dominant implies $|\lambda_1(W)| > |\lambda_i(W)|$, $i = 2,\ldots,n$. Combined with $W\mathbf{1} = \mathbf{1}$, it means that $W = W^\top \in \mathcal{PF}$ or, from Theorem 1, that W is EDS. Viceversa, W EDS means $W \overset{\vee}{>} 0$, i.e., $\rho(W)$ is a simple strictly dominant eigenvalue of W with positive eigenvector. Since it is also $W\mathbf{1} = \mathbf{1}$, from Lemma 1, it must be that $\rho(W) = 1$ is the eigenvalue of multiplicity 1 strictly dominating all other eigenvalues. Hence W is marginally Schur stable. $\square$

Proof of Theorem 6. W ES implies that $W\mathbf{1} = \mathbf{1}$ and, since $W \overset{\vee}{>} 0$, $\lambda_1(W) = \rho(W) = 1$ must be a simple strictly dominant eigenvalue. Hence $|\lambda_i(W)| < 1$, $i = 2,\ldots,n$, meaning that W is also marginally Schur stable. As for the other direction, if W such that $W\mathbf{1} = \mathbf{1}$ is marginally Schur stable with $\lambda(W) = 1$ simple and strictly dominant, it means that $|\lambda_i(W)| < 1$, $i = 2,\ldots,n$, i.e., $\rho(W) = 1$, hence $W \in \mathcal{PF}$. $\square$

Proof of Corollary 3. (i) $\Longleftrightarrow$ (ii) Follows combining Theorem 6 with $W\mathbf{1} = W^\top\mathbf{1} = \mathbf{1}$.
(ii) $\Longleftrightarrow$ (iii) If W is normal, then it can be decomposed as $W = UDU^\top$ where U is orthonormal and D similar to (14), with $\mu_1 = 1$ and $|\mu_i| < 1$, $i = 2,\ldots,\ell$, and $\sqrt{\nu_j^2 + \omega_j^2} < 1$, $j = 1,\ldots,\frac{n-\ell}{2}$. From this, in fact, we have $W^\top W = U(D^\top D)U^\top$, and $I - W^\top W$ has eigenvalues 0, $1 - \mu_i^2 > 0$, $i = 2,\ldots,\ell$, and $1 - (\nu_j^2 + \omega_j^2) > 0$, $j = 1,\ldots,\frac{n-\ell}{2}$, i.e., $I - W^\top W$ is psd of corank 1. $\square$

APPENDIX B
TIME-VARYING DIGRAPHS

Proof of Theorem 7. Denote $\Pi = I - \mathbf{1}\mathbf{1}^\top/n$ the projection matrix onto $\mathrm{span}(\mathbf{1})^\perp$. Then $\mathbf{x}(t)$ can be split as $\mathbf{x}(t) = \Pi\mathbf{x}(t) + (I - \Pi)\mathbf{x}(t) = \delta(t) + \alpha\mathbf{1}$, where $\alpha(t) = \frac{1}{n}\mathbf{1}^\top\mathbf{x}(t)$ is the norm of the projection of $\mathbf{x}(t)$ onto $\mathrm{span}(\mathbf{1})$, and $\delta(t) = \mathbf{x}(t) - \alpha(t)\mathbf{1}$ is the projection onto $\mathrm{span}(\mathbf{1})^\perp$. Here we omit the argument t for simplicity of expression. By (4),

$$\dot\alpha = \frac{1}{n}\mathbf{1}^\top\dot{\mathbf{x}} = -\frac{1}{n}\mathbf{1}^\top L_{\sigma(t)}\mathbf{x} = 0. \tag{15}$$

The third equality is due to the weight balance of each $L_k, k \in [m]$ (which follows from normality of L_k). Therefore, $\alpha(t)$ is invariant, i.e., $\alpha(t) = \alpha(0) = \frac{1}{n}\mathbf{1}^\top\mathbf{x}(0), t \geq 0$.

As for the "disagreement" projection δ, it holds

$$\dot\delta = \dot{\mathbf{x}} = -L_{\sigma(t)}\mathbf{x} = -L_{\sigma(t)}\delta. \tag{16}$$

Define a candidate Lyapunov function $V = \frac{1}{2}\|\delta\|^2 \geq 0$. We have

$$\dot{V} = -\delta^\top\frac{L_{\sigma(t)} + L_{\sigma(t)}^\top}{2}\delta. \tag{17}$$

By Corollary 1, for all $k \in [m]$, $\frac{L_k + L_k^\top}{2}$ is psd of corank 1, and $\mathbf{1}$ is its eigenvector corresponding to the eigenvalue 0. Let

$$\lambda^* = \min_{k\in[m]}\lambda_2\left(\frac{L_k + L_k^\top}{2}\right) > 0.$$

λ^* is real due to the symmetry of $\frac{L_k + L_k^\top}{2}$. By the Courant-Fisher theorem [38], it holds

$$\frac{\dot{V}}{\|\delta\|^2} \leq -\min_{\substack{\mathbf{v}^\top\mathbf{1}=\mathbf{0} \\ \|\mathbf{v}\|=1}}\mathbf{v}^\top\frac{L_{\sigma(t)} + L_{\sigma(t)}^\top}{2}\mathbf{v} = -\lambda^*,$$

i.e., $\dot{V} \leq -\lambda^* V$. This means that V decreases to 0 exponentially at the rate λ^*. Therefore, for any switching signal $\sigma(t)$, $\lim_{t\to\infty}\delta(t) = 0$, i.e., $\lim_{t\to\infty}\mathbf{x}(t) = \alpha(0)\mathbf{1}$. The proof is then completed. $\square$

Proof of Theorem 8. The normality of W_k implies that $W_k^\top\mathbf{1} = \mathbf{1}$. Similarly to the proof of Theorem 7, let $\alpha(t) = \frac{1}{n}\mathbf{1}^\top\mathbf{x}(t), t \in \mathbb{Z}_\geq$. It then holds

$$\alpha(t+1) = \frac{1}{n}\mathbf{1}^\top\mathbf{x}(t+1) = \frac{1}{n}\mathbf{1}^\top W_{\sigma(t)}\mathbf{x}(t) = \alpha(t).$$

Let $\delta(t) = \mathbf{x}(t) - \alpha(t)\mathbf{1}$. We then have

$$\delta(t+1) = W_{\sigma(t)}\mathbf{x}(t) - \alpha(t+1)\mathbf{1} = W_{\sigma(t)}\delta(t).$$

Take a candidate Lyapunov function $V(t) = \delta(t)^\top\delta(t)$. It holds

$$V(t+1) - V(t) = -\delta(t)^\top(I_n - W_{\sigma(t)}^\top W_{\sigma(t)})\delta(t). \tag{18}$$

For $k \in [m]$, since W_k is normal, it can be decomposed as $W_k = UD_kU^\top$, where D_k is as in (14):

$$D_k = \begin{bmatrix} \mu_1^k & & & & & & & \\ & \ddots & & & & & & \\ & & \mu_\ell^k & & & & & \\ & & & \nu_{\ell+1}^k & \omega_{\ell+1}^k & & & \\ & & & -\omega_{\ell+1}^k & \nu_{\ell+1}^k & & & \\ & & & & & \ddots & & \\ & & & & & & \nu_{\frac{n-\ell}{2}}^k & \omega_{\frac{n-\ell}{2}}^k \\ & & & & & & -\omega_{\frac{n-\ell}{2}}^k & \nu_{\frac{n-\ell}{2}}^k \end{bmatrix}$$

Here,

$$\Lambda(W_k) = \{\mu_1^k = 1, \mu_2^k,\ldots,\mu_\ell^k,$$
$$\nu_{\ell+1}^k \pm i\omega_{\ell+1}^k,\ldots,\nu_{\frac{n-\ell}{2}}^k \pm i\omega_{\frac{n-\ell}{2}}^k\}$$

with all the μ^k, ν^k and ω^k real, and U an orthonormal matrix. Therefore, $W_k^\top W_k = U(D_k)^\top D_k U^\top$, with

$$(D_k)^\top D_k = \begin{bmatrix} 1 & & & & & & \\ & (\mu_2^k)^2 & & & & & \\ & & \ddots & & & & \\ & & & (\mu_\ell^k)^2 & & & \\ & & & & \eta_{\ell+1}^k I_2 & & \\ & & & & & \ddots & \\ & & & & & & \eta_{\frac{n-\ell}{2}}^k I_2 \end{bmatrix}$$

with $\eta_j^k = (\nu_j^k)^2 + (\omega_j^k)^2$, $j = \ell+1,\ldots,\frac{n-\ell}{2}$. By Corollary 3,

$$|\mu_j^k| < 1, \quad j = 2,\ldots,\ell$$

and

$$|(\nu_j^k)^2 + (\omega_j^k)^2| < 1, \quad j = \ell + 1, \ldots, \frac{n-\ell}{2}.$$

Then, applying the Courant-Fisher theorem to (18),

$$V(t+1) - V(t) \leq -\tilde{\lambda} V(t),$$

where $\tilde{\lambda} = \max_{k \in [m]} |\lambda_{n-1}(W_k)| < 1$. Therefore,

$$V(t) \leq (1 - \tilde{\lambda})^t V(0),$$

which yields $\lim_{t \to \infty} \delta(t) = 0$. The proof is then completed.
$\square$

Proof of Theorem 9. From the properties of Q,

$$U = \begin{bmatrix} Q \\ \mathbf{1}^\top / \sqrt{n} \end{bmatrix}$$

is an orthonormal change of basis, and

$$U L_k U^\top = \begin{bmatrix} \bar{L}_k & \mathbf{0} \\ \mathbf{0}^\top & 0 \end{bmatrix}, \tag{19}$$

where $\bar{L}_k = Q L_k Q^\top$ is the projection of L_k onto $\mathrm{span}(\mathbf{1})^\perp$. Since $-L_k$ is EEP, it follows from Corollary 1 that it is marginally stable and that its zero eigenvalue has multiplicity 1, while from (19) it follows that L_k and $\bar{L}_k$ share all nonzero eigenvalues. Consequently, $-\bar{L}_k$ is Hurwitz. Considering the projection of P on $\mathrm{span}(\mathbf{1})^\perp$, $\bar{P} = Q P Q^\top$, then it is $\bar{P} = \bar{P}^\top \succ 0$ and

$$-\bar{L}_k \bar{P} - \bar{P} \bar{L}_k^\top = -Q L_k P Q^\top - Q P L_k^\top Q^\top,$$

where we have used that by construction $Q^\top Q = \Pi = I - \mathbf{1}\mathbf{1}^\top/n$, and that $L_k \Pi = L_k$. Existence of a CLF for the projections, i.e., of $\bar{P} = \bar{P}^\top \succ 0$ such that $-\bar{L}_k \bar{P} - \bar{P} \bar{L}_k^\top \prec 0$ $\forall i$, implies that (7) holds, hence, analogously to the proof of Theorem 7, that consensus is achieved for any switching signal $\sigma(t)$.
$\square$

Proof of Theorem 10. From Corollary 3, the projection of W_k onto $\mathrm{span}(\mathbf{1})^\perp$, $\bar{W}_k = Q W_k Q^\top$ is Schur stable. Denoting with $\bar{P} = Q P Q^\top$ the projection of P on $\mathrm{span}(\mathbf{1})^\perp$, then $P \succ 0$ implies $\bar{P} \succ 0$ and

$$\bar{W}_k^\top \bar{P} \bar{W}_k - \bar{P} = Q(W_k^\top P W_k - P)Q^\top \prec 0, \quad k = 1, \ldots, m$$

implies (8). The equality above follows from $\Pi = Q^\top Q = I - \mathbf{1}\mathbf{1}^\top/n$, $W_k \Pi = W_k - \mathbf{1}\mathbf{1}^\top/n$ and $Q\mathbf{1} = 0$.
$\square$

Proof of Proposition 2. Property 1 is true by construction. Property 2 follows from the fact that, since $S^2 = I$, the system (10) corresponds to a change of basis w.r.t. to (2), and consequently Property 3 follows from Lemma 4.
$\square$

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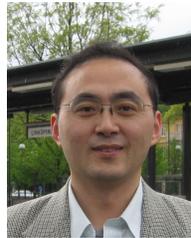
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