On the properties of Laplacian pseudoinverses

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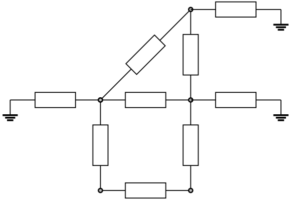
Outline

- Motivating examples
- Properties of Laplacian pseudoinverses of unsigned networks
- Properties of Laplacian pseudoinverses of signed networks
- Application: electrical networks

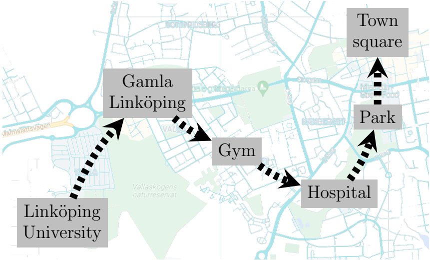
Motivating examples

Problem: Study the properties of Laplacian pseudoinverses

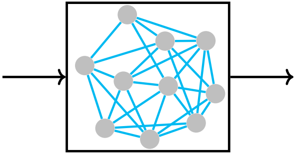
Motivation:



effective resistance
in electrical networks



hitting/commuting times
for a Markov chain

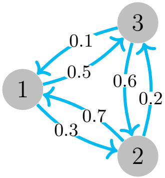


estimate $\mathcal{H}_2$ norm in
networked dynamical systems

Laplacian pseudoinverses of unsigned networks

The Laplacian L of an unsigned graph $\mathcal{G}(A)$ with n nodes is:

$$[L]_{ij} = \begin{cases} \sum_{j=1}^{n} a_{ij}, & j = i \\ -a_{ij}, & j \neq i \end{cases} \quad i, j = 1, \dots, n$$



for each edge (j, i): $a_{ij} > 0$ $\implies$ $L = \begin{bmatrix} 0.8 & -0.7 & -0.1 \\ -0.3 & 0.9 & -0.6 \\ -0.5 & -0.2 & 0.7 \end{bmatrix}$, $L \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = L\mathbb{1} = 0$

and $\mathrm{Re}(\lambda) \geq 0$ for all $\lambda \in \mathrm{sp}(L)$

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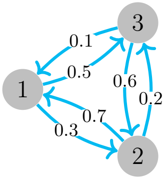

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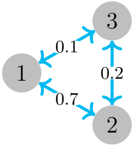
and $\mathrm{Re}(\lambda) \geq 0$ for all $\lambda \in \mathrm{sp}(L)$

Focus on: $L^{\dagger}$ = Moore-Penrose pseudoinverse of L

Properties of Laplacian pseudoinverses of unsigned networks

Which properties of the graph Laplacian L does the Laplacian pseudoinverse $L^\dagger$ satisfy?

- 0 is eigenvalue of $L^\dagger$
- BUT $L^\dagger$ is NOT an M-matrix*



$$L = \begin{bmatrix} 0.8 & -0.7 & -0.1 \\ -0.7 & 0.9 & -0.2 \\ -0.1 & -0.2 & 0.3 \end{bmatrix}$$

$$\mathrm{sp}(L) = \{0, 0.44, 1.56\}$$

$$L^\dagger = \begin{bmatrix} 0.77 & 0.05 & -0.82 \\ 0.05 & 0.63 & -0.68 \\ -0.82 & -0.68 & 1.50 \end{bmatrix}$$

$$\mathrm{sp}(L^\dagger) = \{0, 0.64, 2.26\}$$

$\Rightarrow$ $L^\dagger$ is NOT a Laplacian matrix...

*M-matrix = nonpositive off-diagonal elements + eigenvalues with nonnegative real part

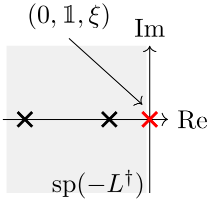
Properties of Laplacian pseudoinverses of unsigned networks

... but $L^\dagger$ still obeys to a strong Perron-Frobenius property:

Properties of Laplacian pseudoinverses of unsigned networks

... but $L^\dagger$ still obeys to a strong Perron-Frobenius property:

- 0 is the "dominant" eigenvalue of $-L^\dagger$
- $\mathbb{1} > 0$ and $\xi > 0$ with $L^\dagger \xi = (L^\dagger)^T \mathbb{1} = 0$



$$L^\dagger = \begin{bmatrix} 0.77 & 0.05 & -0.82 \\ 0.05 & 0.63 & -0.68 \\ -0.82 & -0.68 & 1.50 \end{bmatrix}$$

$$\mathrm{sp}(-L^\dagger) = \{-2.26, -0.64, 0\}$$

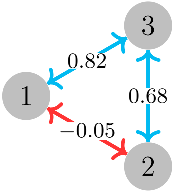
$$L^\dagger \mathbb{1} = (L^\dagger)^T \mathbb{1} = 0$$

Properties of Laplacian pseudoinverses of unsigned networks

The Laplacian pseudoinverse is in a general a signed Laplacian, and:

- 0 is eigenvalue of $L^\dagger$
- $L^\dagger$ and $(L^\dagger)^T$ obey to a Perron-Frobenius property

$$L^\dagger = \begin{bmatrix} 0.77 & 0.05 & -0.82 \\ 0.05 & 0.63 & -0.68 \\ -0.82 & -0.68 & 1.50 \end{bmatrix}$$



signed graph associated with $L^\dagger$

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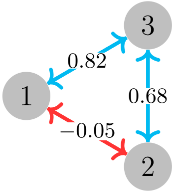


signed graph associated with $L^\dagger$

Can these properties be extended to signed Laplacians, i.e., Laplacians associated to signed graphs?

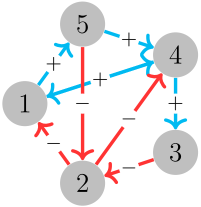
Signed Laplacian matrix

The signed Laplacian L of a signed graph $\mathcal{G}(A)$ with n nodes is:

$$[L]_{ij} = \begin{cases} \sum_{j=1}^{n} a_{ij}, & j = i \\ -a_{ij}, & j \neq i \end{cases} \qquad i, j = 1, \ldots, n$$

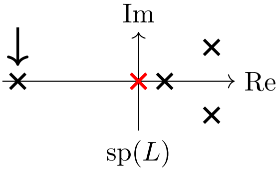
Properties:

- $0 \in \mathrm{sp}(L)$: $L\mathbb{1} = 0$
- L need NOT be diagonally dominant nor an M-matrix

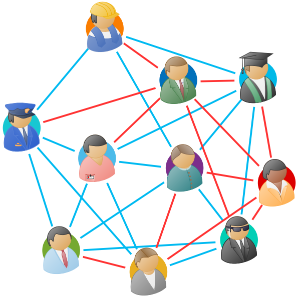


for each edge (j, i): $a_{ij} \lesseqgtr 0$

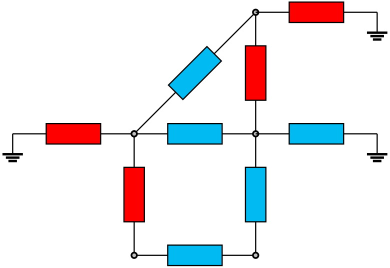
$$L = \begin{bmatrix} \mathbf{0} & +1 & 0 & -1 & 0 \\ 0 & -\mathbf{2} & +1 & 0 & +1 \\ 0 & 0 & +1 & -1 & 0 \\ -1 & +1 & 0 & +1 & -1 \\ -1 & 0 & 0 & 0 & +1 \end{bmatrix}$$



Why signed networks?

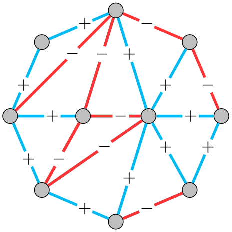


social networks:
friends/enemies



electrical networks:
positive/negative
conductance of resistor

⟹



signed network:
cooperative/antagonistic
interactions

Problem formulation

Problem: Study the algebraic properties of Laplacian pseudoinverses

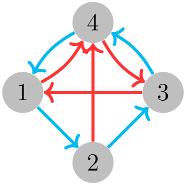
Task: Find a class of (signed) Laplacian matrices that:

- has most of the properties of M-matrices
  - strong Perron-Frobenius property
  - eigenvalues with nonnegative real part
- is closed with respect to pseudoinversion

Eventually exponentially positive (EEP) Laplacians

Let L be a signed Laplacian

$-L$ is eventually exponentially positive: $\exists\, t_0 \in \mathbb{N}$ s.t. $e^{-Lt} > 0$ for all $t \geq t_0$



$$L = \begin{bmatrix} 2.3 & 0 & 1 & -3.3 \\ -2.1 & 2.1 & 0 & 0 \\ 0 & -1.2 & 0.6 & 0.6 \\ 3 & 0.8 & -3.3 & -0.5 \end{bmatrix}, \quad e^{-L\cdot 5} = \begin{bmatrix} 0.12 & 0.28 & 0.52 & 0.08 \\ 0.08 & 0.27 & 0.56 & 0.10 \\ 0.06 & 0.25 & 0.55 & 0.14 \\ 0.12 & 0.26 & 0.48 & 0.14 \end{bmatrix}$$

C. Altafini, "Investigating stability of Laplacians on signed digraphs via eventual positivity", IEEE 58th CDC, 2019.

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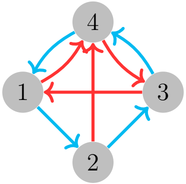

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$\mathrm{sp}(-L) = \{-3.4, -0.6 \pm i3.6, 0\}$ and $L\mathbb{1} = L^T\xi = 0$, $\xi = [0.13\ 0.42\ 0.87\ 0.20]^T$

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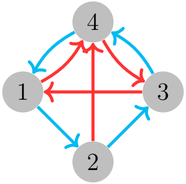
$\iff$ $-L$ and $-L^T$ obey to a strong Perron-Frobenius property

$\implies$ $-L$ is marginally stable of corank 1 ("$\iff$", if L is weight balanced $L\mathbb{1} = L^T\mathbb{1} = 0$)



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Is the class of EEP Laplacians closed w.r.t. pseudoinversion?

Let L be a weight balanced signed Laplacian: $L\mathbb{1} = L^T\mathbb{1} = 0$ ($\iff L^\dagger$ weight balanced)

$-L$ is eventually exponentially positive

$\iff -L^\dagger$ is eventually exponentially positive

$\iff -L^\dagger$ is marginally stable of corank 1

$$L = \begin{bmatrix} 0.15 & 0 & 0 & -0.15 \\ -0.23 & 0.15 & 0.15 & -0.07 \\ 0.01 & -0.12 & -0.03 & 0.14 \\ 0.07 & -0.03 & -0.12 & 0.08 \end{bmatrix} \qquad L^\dagger = \begin{bmatrix} 2.25 & -1.86 & -0.22 & -0.17 \\ -1.60 & 1.51 & -5.67 & 5.75 \\ 2.10 & 0.55 & 4.44 & -7.09 \\ -2.75 & -0.20 & 1.45 & 1.50 \end{bmatrix}$$

$$L\mathbb{1} = L^T\mathbb{1} = 0 \qquad L^\dagger\mathbb{1} = (L^\dagger)^T\mathbb{1} = 0$$

$-L$ EEP $\qquad -L^\dagger$ EEP

$$\mathrm{sp}(-L) = \{-0.17, -0.09 \pm i0.2, 0\} \qquad \mathrm{sp}(-L^\dagger) = \{-5.92, -1.89 \pm i4.24, 0\}$$

Is the class of EEP Laplacians closed w.r.t. symmetrization?

Let L be a normal signed Laplacian: $LL^T = L^T L$ ($\iff L^\dagger$ normal)

$-L$ is eventually exponentially positive

$\iff L_s := \frac{L+L^T}{2}$ is positive semidefinite of corank 1

$\iff L_s^\dagger := \frac{L^\dagger+(L^\dagger)^T}{2}$ is positive semidefinite of corank 1

$$L = \begin{bmatrix} 1.13 & 3.96 & -5.09 \\ -5.09 & 1.13 & 3.96 \\ 3.96 & -5.09 & 1.13 \end{bmatrix}$$

normal, $-L$ EEP, $\mathrm{sp}(L) = \{0, 1.7 \pm i7.8\}$

$$L\mathbb{1} = L^T\mathbb{1} = 0$$

$$L_s = \begin{bmatrix} 1.13 & -0.57 & -0.57 \\ -0.57 & 1.13 & -0.57 \\ -0.57 & -0.57 & 1.13 \end{bmatrix}$$

$\mathrm{sp}(L_s) = \{0, 1.7, 1.7\}$

$$L_s\mathbb{1} = 0$$

Application: Electrical networks

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Electrical network $= \mathcal{G}(A)$ with $A = [a_{ij}] = \frac{1}{\text{resistance between } i \text{ and } j} > 0$

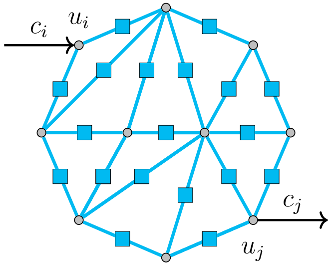
$$L\,u = c$$

graph Laplacian

electric potentials at nodes

current injected into each node, $c^T \mathbb{1} = 0$

$$\Longrightarrow u = L^{\dagger} c + \alpha \mathbb{1}, \quad \alpha \in \mathbb{R}$$



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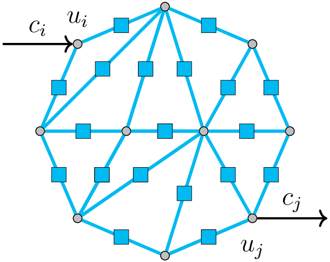
graph Laplacian → L

electric potentials at nodes → u

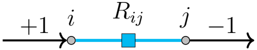
current injected into each node, $c^T \mathbb{1} = 0$ → c

$$\implies u = L^{\dagger} c + \alpha \mathbb{1}, \quad \alpha \in \mathbb{R}$$



If $c_i = +1$, $c_j = -1$ (i.e., $c = e_i - e_j$) then:

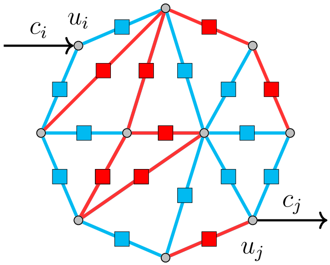
effective resistance between i and j $= R_{ij} = (e_i - e_j)^T L^{\dagger} (e_i - e_j) \geq 0 \quad \forall\, i, j$



Application: Electrical networks

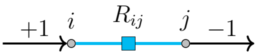
Electrical network $= \mathcal{G}(A)$ with $A = [a_{ij}] = \frac{1}{\text{resistance between } i \text{ and } j} \lessgtr 0$

If signed Laplacian L is normal and $-L$ is EEP then



$$
\begin{aligned}
R_{ij} &= \text{effective resistance between } i \text{ and } j \\
&= (e_i - e_j)^T L^\dagger (e_i - e_j) \\
&= (e_i - e_j)^T \underbrace{\frac{L^\dagger + (L^\dagger)^T}{2}}_{\text{positive semidefinite}} (e_i - e_j)
\end{aligned}
$$

$\Rightarrow \quad R_{ij}$ is well-defined : $R_{ij} \geq 0, \quad R_{ij} = R_{ji} \quad \forall\, i, j$



Conclusion

Problem: Study the algebraic properties of Laplacian pseudoinverses

Class of eventually exponentially positive (EEP) Laplacians is closed

if L is weight balanced, w.r.t.:

- *pseudoinversion*: $-L$ EEP $\iff -L^\dagger$ EEP
- *"stability"*: $-L$ EEP $\iff -L, -L^\dagger$ marginally stable (corank 1)

if L is normal, w.r.t.:

- *symmetrization*: $-L$ EEP $\iff \frac{L+L^T}{2}, \frac{L^\dagger+(L^\dagger)^T}{2}$ positive semidefinite (corank 1)

Application: Electrical networks

Thank you!

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