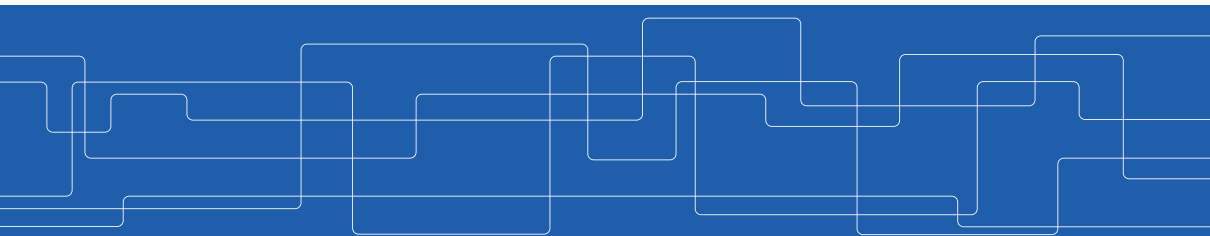




Collective decision-making on networked systems in presence of antagonistic interactions

Angela Fontan, angfon@kth.se
Division of Decision and Control Systems
KTH Royal Institute of Technology, Sweden

in collaboration with
Prof. Claudio Altafini, Linköping University, Sweden





Outline

- ▶ **Background**

Motivation and problem statement

- ▶ **Signed networks**

Notions of structural balance and frustration

- ▶ **Model for collective decision-making over signed networks**

Bifurcation analysis on structurally balanced and structurally unbalanced networks

- ▶ **Application**

Process of government formation over signed “parliamentary networks”

Background

Motivation



Animal groups

⇒ decision reached through **collaboration**

Background

Motivation

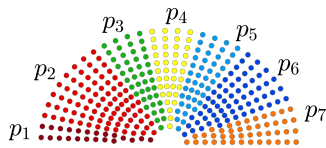


Animal groups

⇒ decision reached through **collaboration**



Social Networks

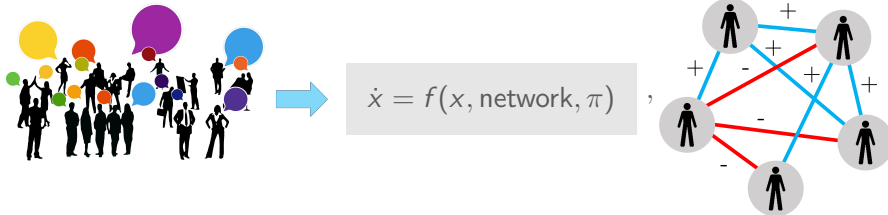


Parliamentary Systems

⇒ both **cooperative** and **antagonistic** interactions may coexist

Background

Problem: collective decision-making in presence of antagonism



1. Signed networks

- positive weight: **cooperative** interaction
- negative weight: **antagonistic** interaction

2. Model for collective decision-making

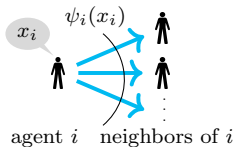
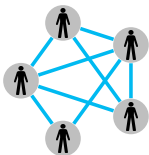
- x : vector of opinions
- equilibrium points: possible decisions

Model for collective decision-making

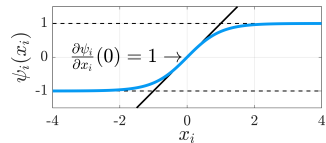
$$\dot{x} = -\Delta x + \pi A \psi(x)$$

- ▶ n agents, $x \in \mathbb{R}^n$ vector of opinions
- ▶ “inertia” of the agents: $\Delta = \text{diag}\{\delta_1, \dots, \delta_n\}$, $\delta_i > 0$
- ▶ interactions between the agents:

unsigned (connected) network $\mathcal{G}(A)$



$$\psi(x) = [\psi_1(x_1) \dots \psi_n(x_n)]^T$$



and $\pi > 0$ scalar parameter

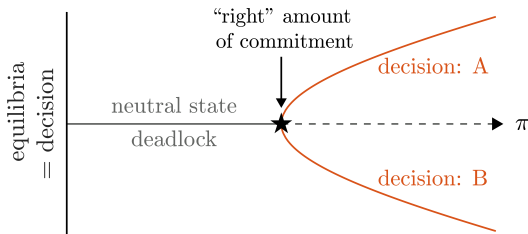
Model for collective decision-making

$$\dot{x} = -\Delta x + \pi A \psi(x) \quad (*)$$

- ▶ π = “social effort” or “strength of commitment” among the agents
- ▶ equilibria = decisions

Assumption: $\delta_i = \sum_j a_{ij} \Rightarrow L = \Delta - A$: **Laplacian** of $\mathcal{G}(A)$

Task: Study qualitative behavior of (*) as social effort parameter π is varied

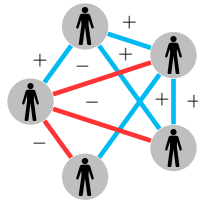


Model for collective decision-making *over signed networks*

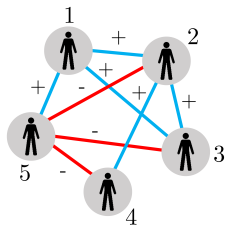
Task: Study the decision-making process in a community of agents where **both cooperative and antagonistic interactions coexist**

Model: $\dot{x} = -\Delta x + \pi A \psi(x)$

Assumptions: $\mathcal{G}(A)$ is **signed**, π : “social effort” between the agents



Signed networks and signed Laplacian matrix

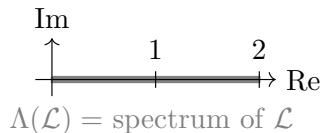


$$A = \begin{bmatrix} 0 & + & + & 0 & + \\ + & 0 & + & + & - \\ + & + & 0 & 0 & - \\ 0 & + & 0 & 0 & - \\ + & - & - & - & 0 \end{bmatrix} \Rightarrow \begin{matrix} \delta_1 \\ \dots \\ \delta_5 \end{matrix} \quad \mathcal{L} = \begin{bmatrix} 1 & - & - & 0 & - \\ - & 1 & - & - & + \\ - & - & 1 & 0 & + \\ 0 & - & 0 & 1 & + \\ - & + & + & + & 1 \end{bmatrix}$$

Signed Laplacian:

$$L = \Delta - A$$

$$\Delta = \text{diag}\{\delta_1, \dots, \delta_n\} : \delta_i = \sum_{j=1}^n |a_{ij}| > 0 \quad \forall i$$



Focus on:

normalized signed Laplacian: $\mathcal{L} = I - \Delta^{-1}A$

Structural balance

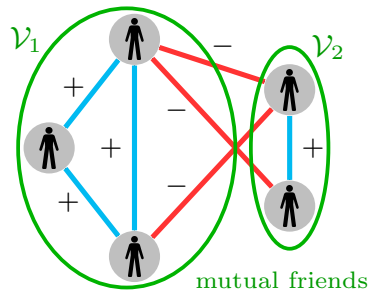
A connected signed graph is

structurally balanced

if $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ s.t. every edge:

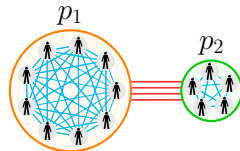
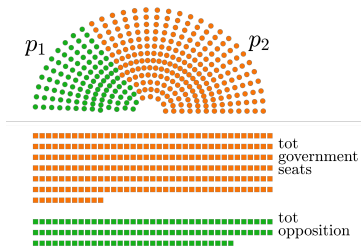
- between \mathcal{V}_1 and \mathcal{V}_2 is negative
- within \mathcal{V}_1 or \mathcal{V}_2 is positive

It is **structurally unbalanced**
otherwise

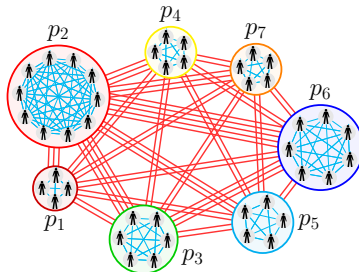
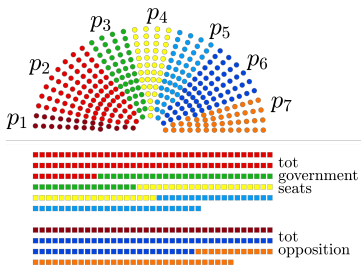


Example: Parliamentary systems

Structurally balanced network



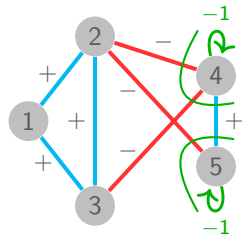
Structurally unbalanced network



Structural balance: equivalent conditions

$\mathcal{G}(A)$ connected signed graph is **structurally balanced** iff

- \exists signature matrix
 $S = \text{diag}\{s_1, \dots, s_n\}$, $s_i = \pm 1$, s.t.
 $S\mathcal{L}S$ has all nonpositive
 off-diagonal entries ($SAS \geq 0$)



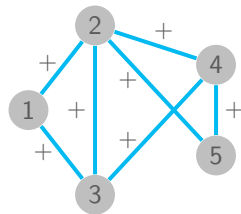
$$A = \begin{bmatrix} 0 & + & + & & \\ + & 0 & + & - & - \\ + & + & 0 & - & \\ - & - & 0 & + & \\ - & & + & 0 & \end{bmatrix} \quad \mathcal{L} = \begin{bmatrix} 1 & - & - & & \\ - & 1 & - & + & + \\ - & & 1 & + & \\ + & + & 1 & - & \\ + & & - & 1 & \end{bmatrix}$$

$$S = \text{diag}\{1, 1, 1, -1, -1\}$$

Structural balance: equivalent conditions

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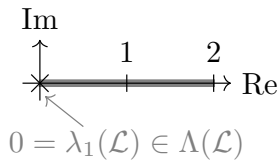
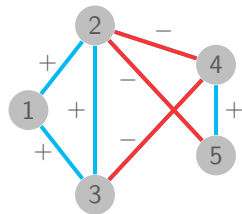
$$S = \text{diag}\{1, 1, 1, -1, -1\}$$

$$SAS = \begin{bmatrix} 0 & + & + & & \\ + & 0 & + & + & + \\ + & + & 0 & + & \\ + & + & + & 0 & + \\ + & & + & + & 0 \end{bmatrix} \quad S\mathcal{L}S = \begin{bmatrix} 1 & - & - & & \\ - & 1 & - & - & - \\ - & - & 1 & - & \\ - & - & - & 1 & - \\ - & & - & - & 1 \end{bmatrix}$$

Structural balance: equivalent conditions

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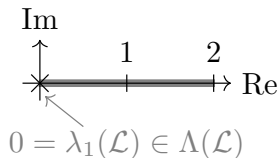
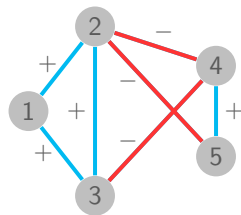
1. \exists signature matrix
 $S = \text{diag}\{s_1, \dots, s_n\}$, $s_i = \pm 1$, s.t.
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2. $\lambda_1(\mathcal{L}) = 0$



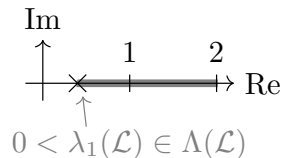
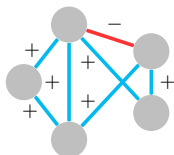
Structural balance: equivalent conditions

$\mathcal{G}(A)$ connected signed graph is **structurally balanced** iff

- \exists signature matrix $S = \text{diag}\{s_1, \dots, s_n\}$, $s_i = \pm 1$, s.t. $S\mathcal{L}S$ has all nonpositive off-diagonal entries ($SAS \geq 0$)
- $\lambda_1(\mathcal{L}) = 0$



$\Rightarrow \mathcal{G}(A)$ connected signed graph is **structurally unbalanced** iff $\lambda_1(\mathcal{L}) > 0$





Frustration index and algebraic conflict

Task: characterize the graph distance from structurally balanced state



Frustration index and algebraic conflict

Task: characterize the graph distance from structurally balanced state

► Frustration Index

(computation: NP-hard problem)

$$\epsilon(\mathcal{G}) = \min_{\substack{S = \text{diag}\{s_1, \dots, s_n\} \\ s_i = \pm 1}} \frac{1}{2} \cdot \underbrace{\sum_{i \neq j} [|\mathcal{L}| + S\mathcal{L}S]_{ij}}_{=e(S): \text{ "energy functional"}}$$

Frustration index and algebraic conflict

Task: characterize the graph distance from structurally balanced state

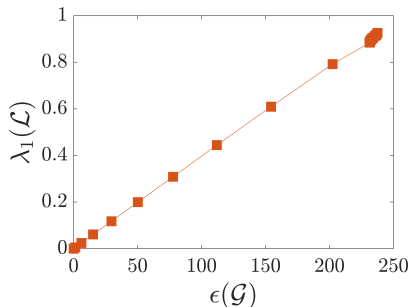
► Frustration Index

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► Algebraic Conflict

$$\xi(\mathcal{G}) = \lambda_1(\mathcal{L})$$



⇒

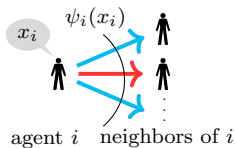
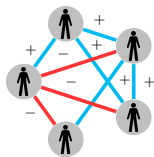
$\lambda_1(\mathcal{L})$ good approximation of $\epsilon(\mathcal{G})$

Model for collective decision-making over signed networks

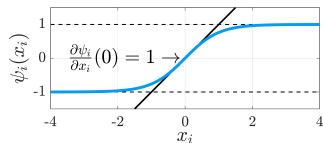
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- ▶ interactions between the agents:

signed (connected) network $\mathcal{G}(A)$



$$\psi(x) = [\psi_1(x_1) \dots \psi_n(x_n)]^T$$



and $\pi > 0$ “social effort” (or “strength of commitment”)

Dynamical interpretation of structural balance

$$\dot{x} = -\Delta x + \pi A \psi(x) = \Delta(-x + \underbrace{\pi \Delta^{-1} A \psi(x)}_{:=H}) \quad (\star)$$

“Laplacian” assumption: $\delta_i = \sum_j |a_{ij}| > 0 \forall i \Rightarrow \mathcal{L} = I - H$

Then at the origin for $\pi = 1$:

$$\text{Jacobian: } J = -L = \Delta(-\mathcal{L})$$

and

$$(\star) \text{ is monotone} \Leftrightarrow \mathcal{G}(A) \text{ is structurally balanced} \Leftrightarrow \lambda_1(\mathcal{L}) = 0.$$



Task

$$\dot{x} = -\Delta x + \pi A\psi(x) = \Delta(-x + \pi H\psi(x)) \quad (\star)$$

Investigate how:

- ▶ the **social effort parameter** π affects the existence and stability of the equilibrium points of the system (\star)
Tool: bifurcation theory ($\mathcal{L} = I - H$ has simple eigenvalues)
- ▶ the presence of **antagonistic** interactions affects the behavior of (\star)
Tool: signed networks theory (frustration)

Bifurcation analysis: structurally balanced networks

$$\dot{x} = \Delta(-x + \pi H\psi(x)), \quad x \in \mathbb{R}^n$$

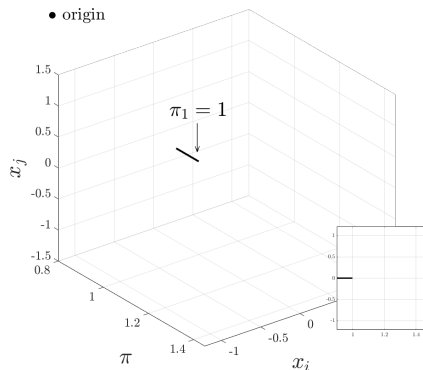
$\pi < 1$: $x = 0$ only eq. point (GAS)

$\pi = 1$: pitchfork bifurcation

- ▶ $x = 0$ saddle point
- ▶ new equilibria: x^* , $-x^*$ (loc. AS $\forall \pi > 1$)

$\pi = \pi_2 = \frac{1}{1-\lambda_2(\mathcal{L})}$: pitchfork bifurcation

- ▶ new equilibria (stable/unstable for $\pi > \pi_2$)



Bifurcation diagram

A. Fontan and C. Altafini, "Multiequilibria analysis for a class of collective decision-making networked systems", IEEE TCNS, 2018

Bifurcation analysis: structurally balanced networks

$$\dot{x} = \Delta(-x + \pi H\psi(x)), \quad x \in \mathbb{R}^n$$

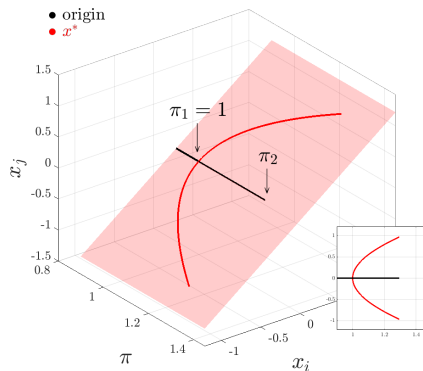
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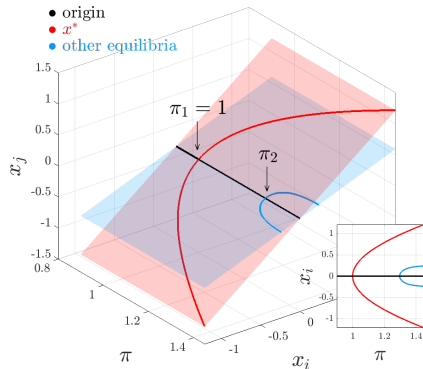
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Bifurcation diagram

A. Fontan and C. Altafini, "Multiequilibria analysis for a class of collective decision-making networked systems", IEEE TCNS, 2018

Bifurcation analysis: structurally unbalanced networks

$$\dot{x} = \Delta(-x + \pi H\psi(x)), \quad x \in \mathbb{R}^n$$

$$\pi_1 = \frac{1}{1 - \lambda_1(\mathcal{L})} \quad \pi_2 = \frac{1}{1 - \lambda_2(\mathcal{L})}$$

$\pi < \pi_1$: $x = 0$ only eq. point (GAS)

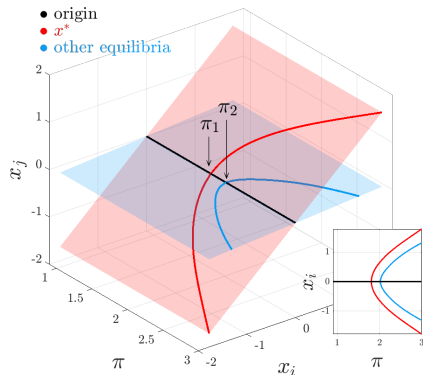
$\pi = \pi_1$: pitchfork bifurcation

▶ $x = 0$ saddle point

▶ new equilibria: x^* , $-x^*$ (loc. AS)

$\pi = \pi_2$: pitchfork bifurcation

▶ new equilibria (stable/unstable for $\pi > \pi_2$)



Bifurcation diagram

A. Fontan and C. Altafini, "The role of frustration in collective decision-making dynamical processes on multiagent signed networks", IEEE TAC, 2022.

Sketch of the proof: first bifurcation

Theorem

Assuming:

- ▶ S-shaped ψ : $\forall i$ ψ_i is odd, saturated, sigmoidal, monotonically increasing with $\frac{\partial \psi_i}{\partial x_i}(0) = 1$
- ▶ $\lambda_1(\mathcal{L}) > 0$ simple

Then:

$$x^* \neq 0 \text{ is equilibrium point of } \dot{x} = \Delta(-x + \pi H\psi(x)) \iff \pi > \pi_1 = \frac{1}{1 - \lambda_1(\mathcal{L})}$$

Sketch of the proof: first bifurcation

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Proof: Sufficiency [$x = 0$ is GAS when $\pi \leq \pi_1$]

Lyap. function $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$, $V(x) = \sum_i \int_0^{x_i} \psi_i(s) ds \geq 0$ (radially unbounded)

$$\begin{aligned} \dot{V}(x) &= \psi(x)^T \dot{x} = \underbrace{-\psi(x)^T \Delta x}_{> \psi(x)^T \Delta x} + \underbrace{\psi(x)^T \Delta(\pi H)\psi(x)}_{= \Delta^{\frac{1}{2}}(\pi \Delta^{\frac{1}{2}} H \Delta^{-\frac{1}{2}}) \Delta^{\frac{1}{2}}} \\ &< \underbrace{-\psi(x)^T \Delta^{\frac{1}{2}} (I - \pi \Delta^{\frac{1}{2}} H \Delta^{-\frac{1}{2}}) \Delta^{\frac{1}{2}} \psi(x)}_{\text{symmetric, psd } (\geq 0)} \leq 0 \quad \forall x \neq 0 \end{aligned}$$

Sketch of the proof: first bifurcation

Proof: Necessity [pitchfork bifurcation when $\pi = \pi_1 = \frac{1}{1-\lambda_1(L)} = \frac{1}{\lambda_n(H)}$]

$$\Phi(x, \pi) = -x + \pi H\psi(x) = 0, \quad J := \frac{\partial \Phi}{\partial x}(0, \pi_1) = -I + \pi_1 H$$

Lyapunov-Schmidt reduction:

- ▶ v (right), w (left) eigenvectors of J relative to 0 \Rightarrow

$$\begin{aligned} E = I - vw^T &: \mathbb{R}^n \rightarrow \text{range}(J) \\ I - E &: \mathbb{R}^n \rightarrow \ker(J) \end{aligned}$$
- ▶ split $x = yv + r$, $y \in \mathbb{R}$ and $r = Ex \Rightarrow$ near $(0, \pi_1)$:

$$\begin{cases} 0 = E \Phi(yv + r, \pi) \\ 0 = (I - E) \Phi(yv + r, \pi) \end{cases}$$
- ▶ implicit function theorem: $\exists! r = R(yv, \pi) : E \Phi(yv + R(yv, \pi), \pi) = 0$
- ▶ define center manifold $g : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ by: $g(y, \pi) := w^T (I - E) \Phi(yv + R(yv, \pi), \pi)$
- ▶ partial derivatives at $(0, \pi_1)$ satisfy

$$g_y = g_{yy} = g_\pi = 0, \quad g_{\pi y} > 0, \quad g_{yyy} < 0 \quad \Rightarrow \quad \text{pitchfork bifurcation at } \pi = \pi_1!$$

□

Interpretation of the results.. as we vary π

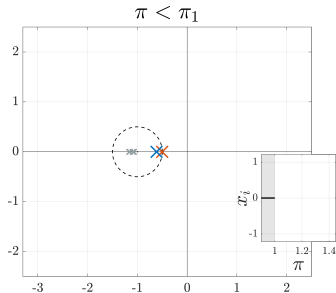
Pitchfork bifurcation at: $\pi_1 = \frac{1}{1-\lambda_1(\mathcal{L})}$, $\pi_2 = \frac{1}{1-\lambda_2(\mathcal{L})}$

(norm.) linearization at 0:
 $-I + \pi H = -I + \pi(I - \mathcal{L})$

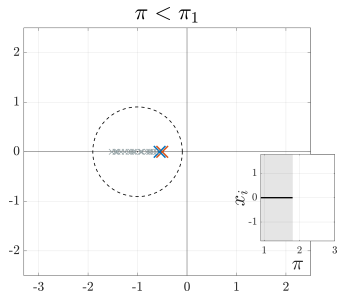
× $\lambda_1(-I + \pi(I - \mathcal{L}))$

× $\lambda_2(-I + \pi(I - \mathcal{L}))$

structurally balanced



structurally unbalanced



$\pi < \pi_1$

(small effort)

■ no decision (deadlock)

Interpretation of the results.. as we vary π

Pitchfork bifurcation at: $\pi_1 = \frac{1}{1-\lambda_1(\mathcal{L})}$, $\pi_2 = \frac{1}{1-\lambda_2(\mathcal{L})}$

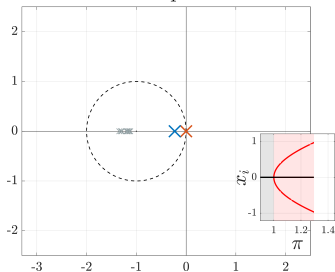
(norm.) linearization at 0:
 $-I + \pi H = -I + \pi(I - \mathcal{L})$

× $\lambda_1(-I + \pi(I - \mathcal{L}))$

× $\lambda_2(-I + \pi(I - \mathcal{L}))$

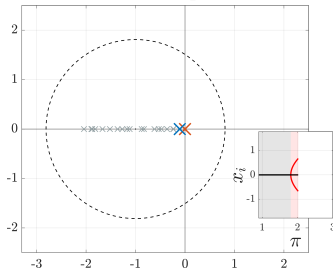
structurally balanced

$\pi = \pi_1 = 1$



structurally unbalanced

$\pi = \pi_1$



$\pi < \pi_1$

(small effort)

■ no decision (deadlock)

$\pi \in (\pi_1, \pi_2)$

“right” commitment

■ two (alternative) decisions

Interpretation of the results.. as we vary π

Pitchfork bifurcation at: $\pi_1 = \frac{1}{1-\lambda_1(\mathcal{L})}$, $\pi_2 = \frac{1}{1-\lambda_2(\mathcal{L})}$

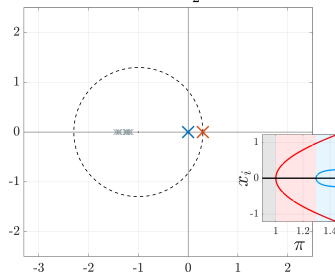
(norm.) linearization at 0:
 $-I + \pi H = -I + \pi(I - \mathcal{L})$

× $\lambda_1(-I + \pi(I - \mathcal{L}))$

× $\lambda_2(-I + \pi(I - \mathcal{L}))$

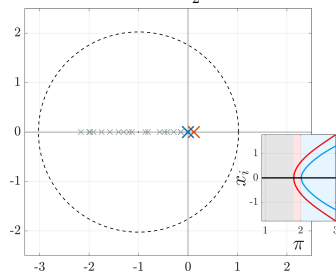
structurally balanced

$\pi = \pi_2$



structurally unbalanced

$\pi = \pi_2$



$\pi < \pi_1$

(small effort)

■ no decision (deadlock)

$\pi \in (\pi_1, \pi_2)$

“right” commitment

■ two (alternative) decisions

$\pi > \pi_2$

“overcommitment”

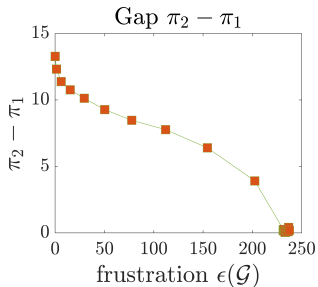
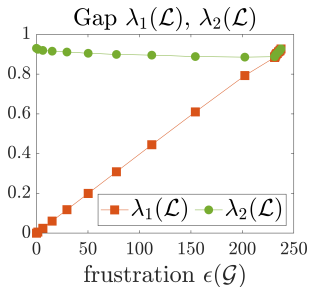
■ several decisions

Interpretation of the results.. as we vary the frustration

Signed network \mathcal{G} with frustration $\epsilon(\mathcal{G})$

$$\pi_1 = \frac{1}{1 - \lambda_1(\mathcal{L})} \begin{cases} = 1 \text{ fixed,} & \text{structurally balanced } \mathcal{G} \\ \text{depends on } \epsilon(\mathcal{G}), & \text{structurally unbalanced } \mathcal{G} \end{cases}$$

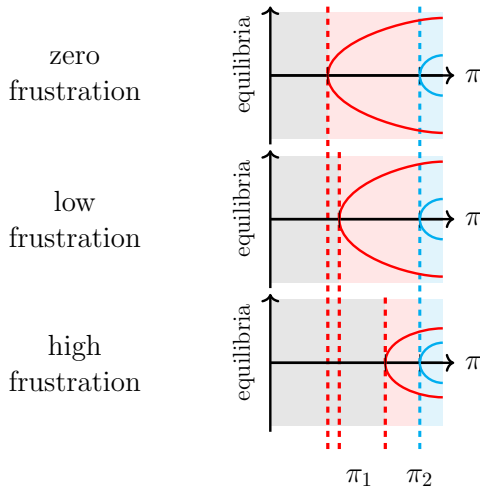
$$\pi_2 = \frac{1}{1 - \lambda_2(\mathcal{L})} \begin{cases} \text{depends on algebraic connectivity,} & \text{structurally balanced } \mathcal{G} \\ \text{independent from } \epsilon(\mathcal{G}), & \text{structurally unbalanced } \mathcal{G} \end{cases}$$



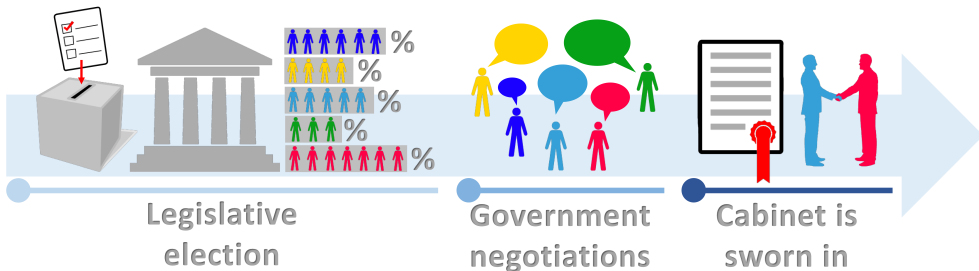
Summary

- ▶ $\pi_1 = \frac{1}{1-\lambda_1(\mathcal{L})}$ grows with $\lambda_1(\mathcal{L})$
- ▶ $\lambda_1(\mathcal{L}) \approx$ frustration
- ▶ the higher the frustration:
 - the higher the social effort needed to achieve a decision
 - the smaller the interval for which only two alternative decisions exist

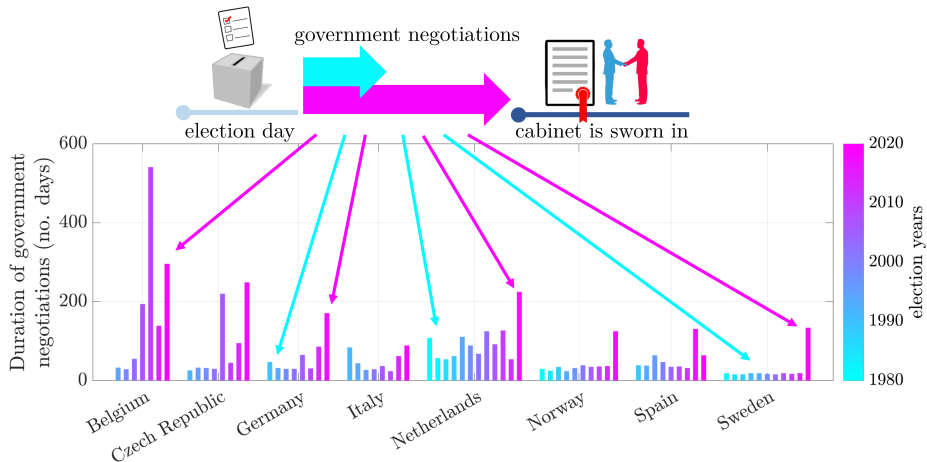
SIGNED GRAPH DYNAMICAL SYSTEM



Government formation in parliamentary democracies



Duration of government negotiation phase



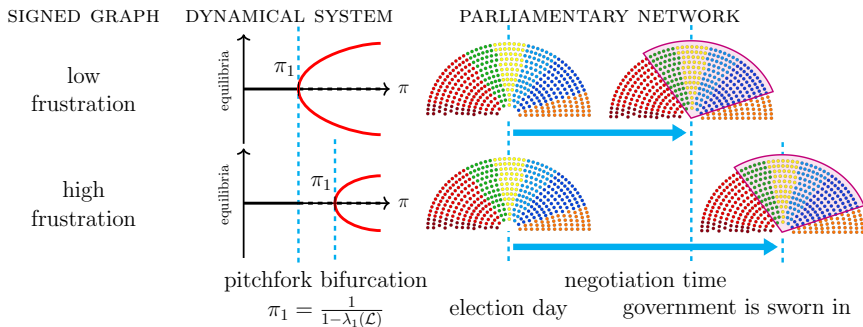
Question: can we use our model to explain this behavior?

Dynamics of the formation of a government

- ▶ signed network: **parliament**
- ▶ decision: **vote of confidence** of the parliament
- ▶ social effort: **duration** of the government negotiation phase

$$\lambda_1(\mathcal{L}) \sim \text{frustration} \quad + \quad \pi_1 \sim \text{duration of negotiations} \quad + \quad \pi_1 = \frac{1}{1-\lambda_1(\mathcal{L})}$$

$$\Rightarrow \text{duration of negotiations} \sim \text{frustration}$$

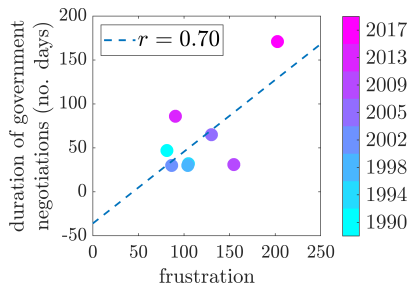
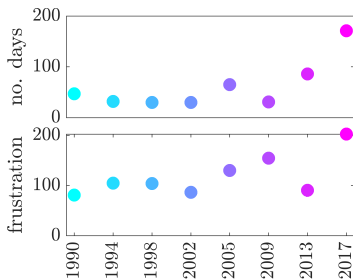


Frustration vs duration of government negotiations

Task: show that the government formation process is influenced by the frustration of the parliamentary network

- ▶ Data: elections in 29 European countries (election years: 1978 - 2020)
- ▶ Method: Pearson's correlation index (r), frustration vs duration of negotiations

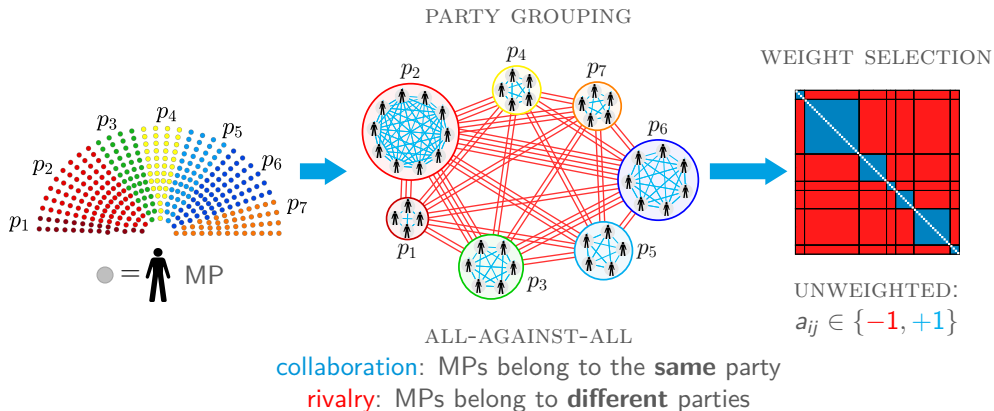
Example: German elections



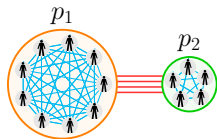
A. Fontan and C. Altafini, "A signed network perspective on the government formation process in parliamentary democracies", Scientific Reports, 2021

Construction of the parliamentary networks

Definition: complete, undirected, signed graph in which each MP is a node

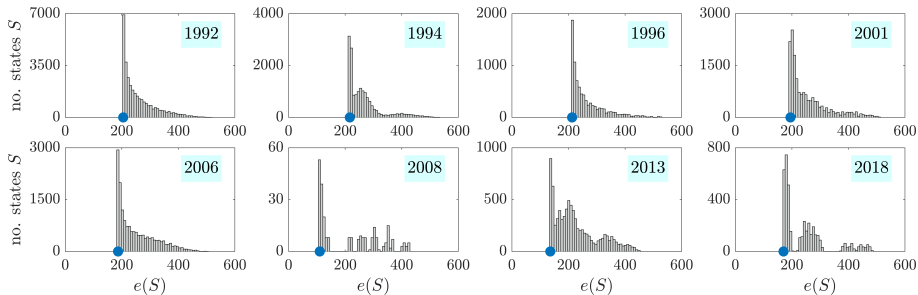


Are the parliamentary networks structurally balanced?



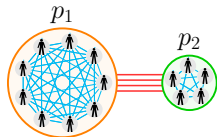
Structurally balanced
parliamentary network

The parliamentary networks have (in general) nonzero frustration..



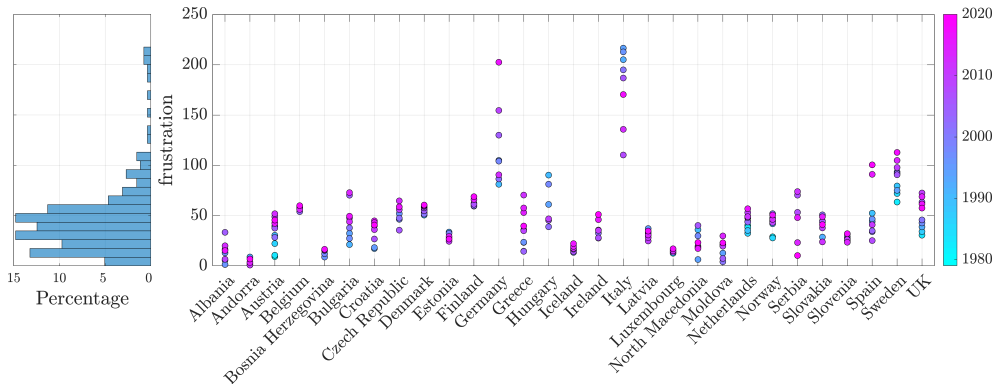
Energy landscape of the Italian parliamentary elections:
● = frustration

Are the parliamentary networks structurally balanced?



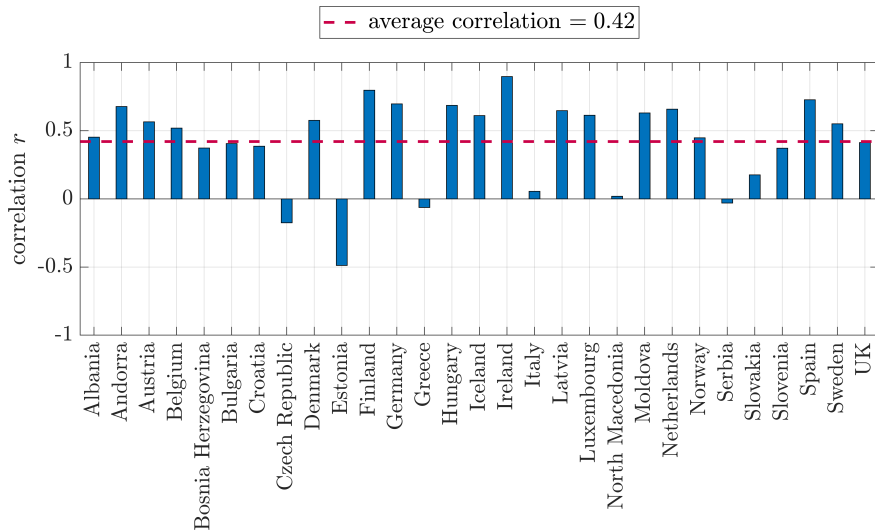
Structurally balanced parliamentary network

The parliamentary networks have (in general) nonzero frustration..

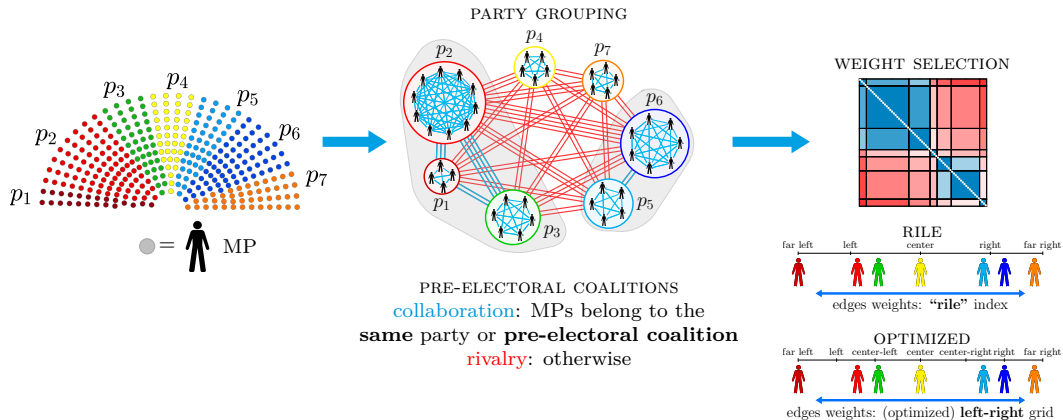


Correlation for all 29 European countries

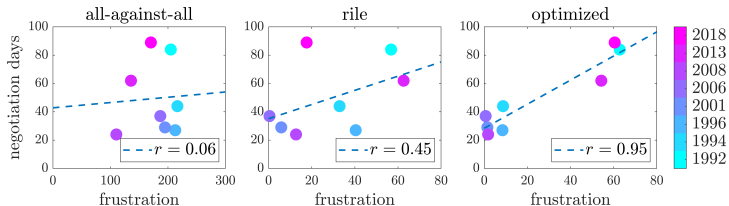
Duration of the government negotiations vs frustration of the parliamentary networks



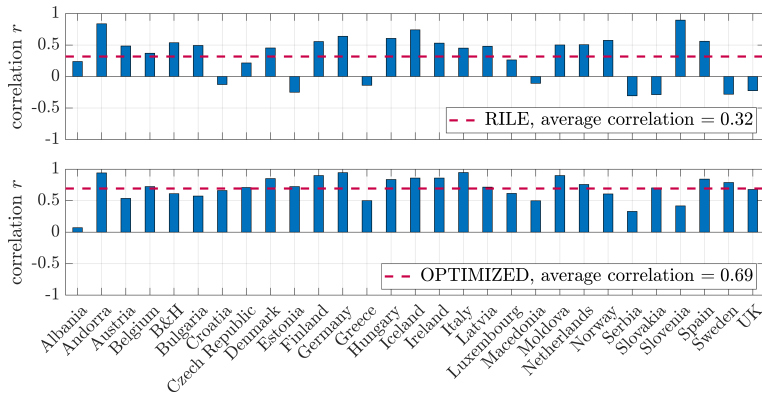
Coalitions and ideological differences in the networks



Correlation for all 29 European countries



Example:
Italian
elections



Results on
average correlation:
0.42, 0.32, 0.69



Conclusion

Task: Study the decision-making process in a community of agents where **both cooperative and antagonistic interactions coexist**

As we vary the **social effort**: pitchfork bifurcation behavior

- ▶ “right” commitment: 2 alternative decisions
- ▶ “overcommitment”: several (more than 2) alternative decisions

As we vary the **frustration** (i.e., amount of disorder) of the signed networks

- ▶ frustration influences the level of commitment required from the agents to reach a decision

Application: Government formation process

- ▶ frustration correlates well with duration of government negotiation phase



Thanks!

Angela Fontan

`angfon@kth.se`

`https://www.kth.se/profile/angfon`

`https://angelafontan.github.io/`